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Jean-Paul Brasselet
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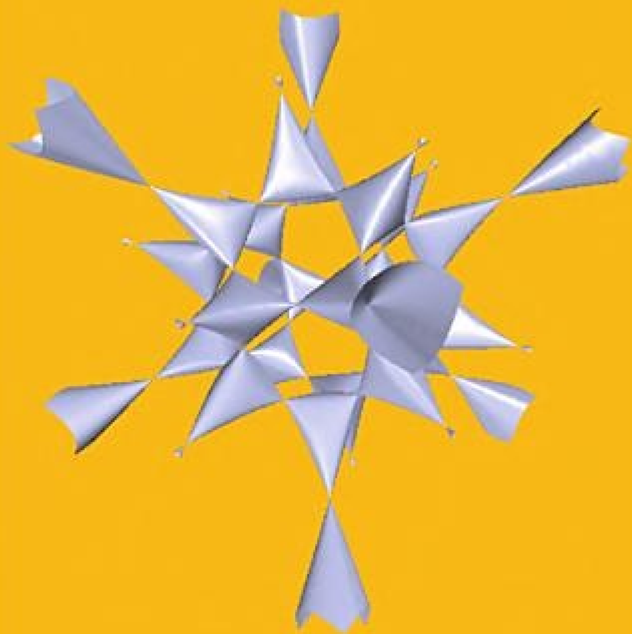
Vector fields on Singular Varieties



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Preface

Vector fields on manifolds play major roles in mathematics and other sciences. In particular, the Poincaré–Hopf index theorem and its geometric counterpart, the Gauss–Bonnet theorem, give rise to the theory of Chern classes, key invariants of manifolds in geometry and topology.

One has often to face problems where the underlying space is no more a manifold but a singular variety. Thus it is natural to ask what is the "good" notion of index of a vector field, and of Chern classes, if the space acquires singularities. The question was explored by several authors with various answers, starting with the pioneering work of M.-H. Schwartz and R. MacPherson.

We present these notions in the framework of the obstruction theory and the Chern–Weil theory. The interplay between these two methods is one of the main features of the monograph.

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We would like to express our indebtedness to M.-H. Schwartz for her great contribution to the theory, that has played and keeps playing an important role in our mathematics.

Contents

Introduction

The study of vector fields and flows near an isolated singularity, or stationary point, has played for decades, and even centuries, a major role in several areas of mathematics and in other sciences, notably in physics, biology, economics, etc. The most basic invariant of a vector field at an isolated singularity is its local Poincaré–Hopf index, which has been studied from very many different viewpoints and there is a vast literature about it. At the same time, it is becoming more and more usual to face problems and situations in Mathematics (and in other sciences) where the underlying space is not a manifold but a singular variety. It is thus natural to ask what should be the "good" notion of index of a vector field on a singular variety, depending on which properties of the local index we have in mind.

For instance one has the theorem of Poincaré–Hopf saying that the sum of the local indices of a vector field with isolated singularities on a closed oriented manifold, is independent of the choice of the vector field and equals the Euler–Poincaré characteristic of the manifold in question. Defining an index for vector fields on singular varieties that has this property leads to the Schwartz index, that we explain below.

Similarly, an important property of the local Poincaré–Hopf index is that it is stable under perturbations, or in other words, that if we approximate the given vector field by another vector field which has only Morse singularities, then the local index of the initial vector field is the number of singularities of its morsification counted with signs. Defining an index for vector fields on singular varieties that has this property leads to a different index, the GSV index.

There are other important properties of the local Poincaré–Hopf index that give rise to various other indices when we look at singular varieties. That makes the study of indices of vector fields over singular varieties an interesting field of current research, which combines an amazing variety of ideas and techniques coming from algebraic topology, differential geometry, algebraic geometry, dynamical systems, mathematical physics, etc.

The goal of this monograph is to give an account of the various indices of vector fields on singular varieties that are in the literature, the relations among them, and the way how they relate with various generalizations of

Chern classes to singular varieties. Indices of vector fields and Chern classes of vector bundles are nowadays present in many branches of mathematics, and these two concepts are linked together in an essential way.

This monograph goes together with [28] to give a global view of the theory of indices and Chern classes for singular spaces. In [28] the focus is on the theory of characteristic classes for singular varieties. Here the emphasis is on indices and their relation with Chern classes. We do this following two of the classical viewpoints for studying Chern classes, both introduced by Chern himself. These are the topological viewpoint, thinking of Chern classes as being the primary obstruction to constructing sections of appropriate fiber bundles, and the differential-geometric viewpoint, via Chern–Weil theory, where the corresponding classes are localized at the "singularities" of certain connections via the theory of residues, which is largely indebted to R. Bott.

The interplay between these two viewpoints for studying indices and characteristic classes, obstruction theory and Chern–Weil theory, is a key feature of this monograph.

This work does not pretend to be comprehensive, and yet it offers a global viewpoint of the theory of indices of vector fields and Chern classes of singular varieties that can be of interest for people working in singularities, algebraic and differential geometry, algebraic topology, and even in string theory and mathematical physics. In each individual chapter we indicate additional references to the literature, for further reading.

The study of indices of vector fields and Chern classes for singular varieties started in the early 1960s with M.-H. Schwartz, and then continued by R. MacPherson and many others. This is today an active field of research, in which the foundations of the theory are being laid out by several authors, and so are their relations with other branches of geometry, topology, and singularity theory.

We start Chap. 1 with the basic, well-known, theory of indices of vector fields and Chern classes that we need in the sequel, and we describe for manifolds the two viewpoints that we use in the rest of the work to study these invariants, namely localization via obstruction theory and localization via Chern–Weil theory.

In Chap. 2 begins the discussion of indices of vector fields on singular varieties. We start with the index introduced by M.-H. Schwartz (in [139,141]) in her study of Chern classes for singular varieties. For her purpose there was no point in considering vector fields in general, but only a special class of vector fields (and frames) that she called "radial," which are obtained by the important process of *radial extension* that she introduced. The generalization of this index to other vector fields was first done by H. King and D. Trotman in [96], and later independently in [6,49,149]. We call this the *Schwartz index*; in the literature it is also called "radial index" because it measures how far the vector field is from being radial. The corresponding discussion for frames is done in Chap. 10.

As mentioned above, one of the basic properties of the local index of Poincaré–Hopf is that it is stable under perturbations. If we now consider an analytic variety V defined, say, by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at 0, and if v is a vector field on V, nonsingular away from 0, then one would like "the index" of v at 0 to be stable under small perturbations of both, the function f and the vector field v. This leads naturally to another concept of index, now called the *GSV index,* introduced in [71, 144, 149], and this is the topic we envisage in Chap. 3.

This monograph mostly concerns vector fields on complex analytic varieties; nevertheless, it is of course interesting to consider also the real analytic case. This is what we do in Chap. 4, where we present the work of M. Aguilar, J. Seade and A. Verjovsky in [6]. That chapter begins with a definition of the Schwartz index in this setting, done independently in [6, 49, 96]. Then we discuss the GSV index, which is now an integer if the singular variety V is odd-dimensional, and an integer modulo 2 if the dimension of V is even. The information one gets is related to previous work by M. Kervaire, C.T.C. Wall and others, and provides an extension of the concept of Milnor number for real analytic map-germs with isolated singularities which may not be algebraically isolated. The viewpoint considered in this chapter is topological; indices of vector fields on real analytic varieties are also considered in Chap. 7 from an algebraic viewpoint, following the work of X. Gómez-Mont et al.

Chapter 5 concerns the virtual index, introduced in [111] by D. Lehmann, M. Soares and T. Suwa for holomorphic vector fields on complex analytic varieties; the extension to continuous vector fields was done in [31, 149]. This index is defined via Chern–Weil theory. The idea comes from the fact that a vector field with an isolated singularity provides a localization of the top dimensional Chern class at the singular point of the vector field, and the number one gets is the corresponding local index of Poincaré–Hopf. Similarly, if $(V, 0)$ is an isolated complete intersection singularity germ in $\mathbb{C}^{n+k}$, an ICIS for short, defined by functions $f = (f_1, \ldots, f_k)$, then a tangent vector field on V, with an isolated singularity, together with the gradient vector fields of the f_i, defines a localization at 0 of the nth Chern class of the ambient space, and the number one gets is *the virtual index* of the vector field. In this context the virtual index coincides with the GSV index, however the definition of the virtual index actually extends to the general setting of vector fields with compact singular set defined on complex analytic varieties which are "strong" local complete intersections.

The previous indices are all defined for continuous vector fields on singular varieties. However, in many situations the vector fields in question are actually analytic, and this is the setting we envisage in Chaps. 6 and 7.

If the vector field is holomorphic, the localization theory via Chern–Weil becomes richer because of the Bott vanishing theorem, producing further interesting residues; this is the topic we study in Chap. 6. A holomorphic vector field defines by integration a one-dimensional holomorphic foliation, and the theory of residues can be developed for general singular holomorphic foliations on certain singular varieties. We consider here one dimensional singular holomorphic foliations, and we refer to [156] for a systematical treatment of the general case. We have three types of residues that arise from a Bott type vanishing theorem: (i) generalizations of Baum–Bott residues to singular varieties, first introduced in [13, 14]; (ii) the Camacho–Sad index, introduced in [42] and used to prove the separatrix theorem. Nowadays there are many generalizations of this index; (iii) variations, introduced in [93] and generalized in [113] (see also [39, 40]). For a local separatrix of a holomorphic vector field, the variation equals the sum of the GSV and the Camacho–Sad indices (see Chap. 6 or Proposition 5 in [40]).

Another remarkable property of the index of Poincaré–Hopf is that, in the case of a germ of holomorphic vector field $v = \sum_{i=1}^{n} h_i \frac{\partial}{\partial z_i}$ on $\mathbb{C}^n$ with an isolated singularity at 0, the local index equals the integer:

$$\dim_{\mathbb{C}} \mathcal{O}_n \big/ (h_1, \ldots, h_n)\,,$$

where $(h_1, \ldots, h_n)$ is the ideal generated by the components of v in the ring $\mathcal{O}_n$ of germs of holomorphic functions at 0 in $\mathbb{C}^n$. This and other facts motivated the search for algebraic formulas for the index of vector fields on singular varieties. The *homological index* of X. Gómez-Mont [68] is a answer to that search. It considers an isolated singularity germ $(V, 0)$ of arbitrary dimension, and a holomorphic vector field on V, singular only at 0. One has the Kähler differentials on V, and a Koszul complex $(\Omega^{\bullet}_{V,0}, v)$:

$$0 \longrightarrow \Omega^{n}_{V,0} \longrightarrow \Omega^{n-1}_{V,0} \longrightarrow \cdots \longrightarrow \mathcal{O}_{V,0} \longrightarrow 0\,,$$

where the arrows are given by contracting forms by the vector field v. The homological index of v is defined to be the Euler characteristic of this complex. If the ambient space V is smooth at 0, the complex is exact in all dimensions, except in degree 0 where the corresponding homology group has dimension equal to the local Poincaré–Hopf index of v at 0. If $(V, 0)$ is an ICIS, the recent article [17] of H.-C. Graf von Bothmer, W. Ebeling and X. Gómez-Mont shows that this index coincides with the GSV index, a fact previously known only for vector fields on hypersurface germs. We remark however that the homological index is defined for vector fields on arbitrary isolated normal singularity germs, while the GSV index is only defined on complete intersection germs. Hence the homological index does provide a new invariant for singular varieties which is not yet understood in general. It would be interesting to know what this index measures globally, *i.e.*, given a compact variety W with isolated singularities and a holomorphic vector field on it with isolated singularities, its total homological index an invariant of W. What type of invariant is it? If W is a local complete intersection, this is just the usual Euler–Poincaré characteristic of a smoothing of W, and as explained in the text, this equals the 0-degree Fulton–Johnson class of W.

In Chap. 7 we briefly describe the homological index, as well as work in this spirit done for real analytic vector fields by X. Gómez-Mont, P. Mardešić and L. Giraldo, generalizing to vector fields on real analytic hypersurface germs the signature formula of Eisenbud–Levin and Khimshiashvili for the local index of real analytic vector fields in $\mathbb{R}^n$.

Chapter 8 concerns the local Euler obstruction, introduced by R. MacPherson in [117] for constructing Chern classes of complex varieties. In [33], J.-P. Brasselet and M.-H. Schwartz defined this invariant via vector fields, interpretation that was essential to prove (also in [33]) that the Schwartz classes of singular varieties coincide with MacPherson's classes via Alexander duality. This viewpoint brings the local Euler obstruction into the framework of "indices of vector fields on singular varieties" and yields to another index, that we may call *the local Euler obstruction* of Whitney stratified vector fields with isolated singularities; the classical local Euler obstruction corresponding to the case of the radial vector field. The Brasselet–Schwartz "Proportionality Theorem" of [33] shows that this index plays an important role when considering liftings of stratified vector fields to sections of the Nash bundle. If the vector field in question comes from the gradient of a function on the singular variety, this local Euler obstruction is the "defect" studied in [32]. By [150], this invariant measures the number of critical points of a local perturbation of the given function which are contained in the regular part of the singular variety, and it is related to several generalizations of the Milnor number to the case of functions on singular varieties.

When considering smooth (real) manifolds, the tangent and cotangent bundles are canonically isomorphic and it does not make much difference to consider either vector fields or 1-forms in order to define their indices and their relations with Chern classes. If the ambient space is a complex manifold, this is no longer the case, but there are still ways for comparing indices of vector fields and 1-forms, and to use these to study Chern classes of manifolds. To some extent this is also true for singular varieties, but there are however important differences and each of the two settings has its own advantages.

R. MacPherson defined the notion of local Euler obstruction in terms of indices of 1-forms on singular varieties. Such indices also appear in the work of C. Sabbah [134,135], particularly in relation with the local Euler obstruction. The systematic study of indices of 1-forms on singular varieties started in a series of articles by W. Ebeling and S. Gusein-Zade This has been, to some extent, a study parallel to the one for vector fields, outlined above. This is the subject of Chap. 9, briefly discussed in this monograph for completeness.

The last part of this work, Chaps. 10–13, concerns several generalizations of Chern classes to the case of singular varieties from the viewpoint of localization theory, by means of indices of vector fields. We refer to [28] for a detailed account on characteristic classes for singular varieties from a global point of view.

In his 1946 original paper [44], S. S. Chern gave several equivalent definitions of his classes, with diverse points of view. In the case of singular

varieties, there are several definitions of characteristic classes, given by various authors. They correspond to various extensions one has of the concept of "tangent bundle" as we go from manifolds to singular varieties. Each of these viewpoints leads to a generalization of Chern classes to the case of singular varieties, as described in [28]. In this monograph we focus on the relations of Chern classes with various indices of vector fields and frames, considering the following four generalizations of the tangent bundle:
(i) the union of the spaces tangent to each stratum of a Whitney stratification of the singular variety;
(ii) the Nash bundle over the Nash blow up of the singular variety;
(iii) the virtual bundle $TM|_V - N|_V$ if the variety V is defined by a holomorphic section of some bundle N over a complex manifold M;
(iv) the tangent sheaf over the singular variety V.

The first generalization is due to M.-H. Schwartz in [139, 141], considering a singular complex analytic variety V embedded in a smooth one M which is equipped with a Whitney stratification adapted to V; she considers a class of stratified frames to define characteristic classes of V which do not depend on M nor on the various choices. These classes live in the cohomology of M with support in V, *i.e.*, $H^*(M, M \setminus V)$. Alexander duality takes this cohomology into the homology of V, and if V is nonsingular the classes one gets in $H_*(V)$ are the homology Chern classes of the manifold, *i.e.*, the Poincaré duals of the usual Chern classes. The 0-dimensional part is the Euler–Poincaré characteristic, which can be localized at the singular set of a vector field and the local contribution is the Schwartz index. The generalization of this index for frames and its relation with Chern classes are given in Chap. 10.

The second extension of the concept of tangent bundle, given by the Nash bundle $\widetilde{T} \to \widetilde{V}$ over the Nash transform $\widetilde{V}$, was used by R. MacPherson [117] to define Chern classes for singular varieties. First one gets the Mather classes of V, also introduced in [117], which are by definition the image of the Chern classes of $\widetilde{T}$ carried into its homology via the Alexander morphism (which is not an isomorphism in general) and then mapped to the homology of V by the morphism $\nu : H_*(\widetilde{V}) \to H_*(V)$. MacPherson's Chern classes for singular varieties [117] live in the homology of V and can be thought of as being the Mather classes of V weighted by the local Euler obstruction in a sense that is made precise in 10.6. MacPherson's classes satisfy important axioms and functoriality properties conjectured by P. Deligne and A. Grothendieck in the early 1970s.

Later, J.-P. Brasselet and M. H. Schwartz proved in [33] that the Alexander isomorphism $H^*(M, M \setminus V) \cong H_*(V)$ carries the Schwartz classes into MacPherson's classes, so they are now called the *Schwartz–MacPherson classes* of V.

The third way of extending the concept of tangent bundle to singular varieties that we mentioned above was introduced by W. Fulton and K. Johnson in [60]. Notice that if a variety $V \subset M$ is defined by a regular section s of a holomorphic bundle E over M, then the bundle $N = E|_V$ is, on the regular part $V_{\rm reg}$, isomorphic to the normal bundle. One has an isomorphism (as C^∞ vector bundles)

$$TM|_{V_{\rm reg}} = TV_{\rm reg} \oplus N|_{V_{\rm reg}} \, ,$$

and therefore the virtual bundle $\tau_V = [TM|_V - N|_V]$, regarded as an element in the K-theory group $KU(V)$, is called the virtual tangent bundle of V. The homology Chern classes of the virtual tangent bundle τ_V are the *Fulton–Johnson classes* of V. In this book we envisage only the case, where V is a local complete intersection in the complex manifold M. When localized at the singular set of a vector field, the local contribution to the 0-dimensional part of the Fulton–Johnson class is the virtual index. When V has only isolated singularities, this corresponds to the Euler–Poincaré characteristic of a smoothing of V. The generalization of the virtual index for frames and its relation with Chern classes are given in Chap. 11.

In general, these classes are different from the Schwartz–MacPherson classes. If V has only isolated singularities, then (by [149,155]) the Schwartz–MacPherson and Fulton–Johnson classes coincide in all dimensions other than 0, and in dimension 0 this difference is given by the local Milnor numbers of V at its singular points. Hence it is natural to call *Milnor classes* the difference between Fulton–Johnson and Schwartz–MacPherson classes. These classes were studied by P. Aluffi [8], who called them μ-classes; there have been significant contributions to the subject afterwards, either by Aluffi himself and by various other authors, such as S. Yokura, A. Parusiński and P. Pragacz, D. Lehmann, T. Ohmoto, J. Schürmann, and the authors of this monograph. This is studied in Chap. 12.

Of course one may also compare Chern–Mather with Fulton–Johnson classes. This was done in [125] for (strong) local complete intersections with isolated singularities, using results of [149,155]. As in the previous case, these classes coincide in all dimensions greater than 0; in dimension 0 their difference is given by the polar multiplicities of T. Gaffney. The corresponding study for varieties with nonisolated singularities has not been done yet.

Finally, the fourth way for extending the concept of tangent bundle to singular varieties by considering the tangent sheaf Θ_V, which is by definition the dual of Ω_V, the sheaf of Kähler differentials on V, fits within the framework considered in [158] of Chern classes for coherent sheaves. We briefly describe some of their properties in Chap. 13. In particular, if V is a local complete intersection in M, then one has a canonical locally free resolution of Ω_V and the corresponding Chern classes essentially coincide with the Fulton–Johnson classes, though the corresponding classes for Θ_V differ from these.

In the sequel we explain how the various indices of vector fields that we discuss in Chaps. 2–8 are related among themselves and how they relate to some generalization of the Chern classes of manifolds to the case of singular varieties. There is however something missing in this picture: so far we do not know of a direct relation between the homological index and some type of Chern classes for singular varieties, neither we know of a direct relation between the Chern classes of the tangent sheaf (or its dual) and some index of vector fields (or 1-forms).

While writing this monograph we have tried to convey the reader a unified view of the various generalizations for singular varieties one has of the important concepts of the local index of Poincaré–Hopf and Chern classes of manifolds. These are topics of current research which keep developing and the literature is vast, so we focused on the most classical approaches for this subject. There are of course important topics that were just glanced here, or maybe even not discussed at all, specially concerning new trends in algebraic geometry and topology, such as string theory and motivic integration. Yet, we think the content of this monograph contributes to lay down the foundations of a theory for singular varieties which is just beginning to be developed and understood. This ought to play in the future such an important role for understanding the geometry and topology of singular varieties as they do for manifolds. And this should also have important applications to other branches of knowledge, where it is important to consider vector fields and flows on orbifolds and singular varieties.

Chapter 1
The Case of Manifolds

Abstract In this chapter we review briefly some of the fundamental results of the classical theory of indices of vector fields and characteristic classes of smooth manifolds. These were first defined in terms of obstructions to the construction of vector fields and frames. In the case of a vector field the Poincaré–Hopf Theorem says that Euler–Poincaré characteristic is the obstruction to constructing a nonzero vector field tangent to a compact manifold. Extension of this result to frames yields to the definition of Chern classes from the viewpoint of obstruction theory.

There is another important point of view for defining characteristic classes on the differential geometry side, this is the Chern–Weil theory. Sections 3 and 4 provide an introduction to that theory and the corresponding definition of Chern classes.

Finally, Sect. 5 sets up one of the key features of this monograph: the interplay between localization via obstruction theory, which yields to the classical relative characteristic classes, and localization via Chern–Weil theory, which yields to the theory of residues. This is one way of thinking of the Poincaré–Hopf Theorem and its generalizations.

Throughout the book, M will denote either a complex manifold of (complex) dimension m, or a C^∞ manifold of (real) dimension m'.

1.1 Poincaré–Hopf Index Theorem

1.1.1 Poincaré–Hopf Index at Isolated Points

Let $v = \sum_{i=1}^{m'} f_i \partial/\partial x_i$ be a vector field on an open set $U \subset \mathbb{R}^{m'}$ with coordinates $\{(x_1, \ldots, x_{m'})\}$. The vector field is said to be continuous, smooth, analytic, etc., according as its components $\{f_1, \ldots, f_{m'}\}$ are continuous, smooth, analytic, etc., respectively (here "smooth" means C^∞, however in most cases C^1 is sufficient). A *singularity* a of v is a point where all of its components vanish, *i.e.*, $f_i(a) = 0$ for all $i = 1, \ldots, m'$. The singularity is *isolated* if at every point x near a there is at least one component of v which is not zero.

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The Poincaré–Hopf index of a vector field at an isolated singularity is its most basic invariant, and it has many interesting properties. To define it, let v be a continuous vector field on U with an isolated singularity at a, and let $\mathbb{S}_\varepsilon$ be a small sphere in U around a. Then the (local) Poincaré–Hopf index of v at a, denoted by $\mathrm{Ind}_{\mathrm{PH}}(v, a)$ (if there is no fear of confusion, we will denote it simply by $\mathrm{Ind}(v, a)$), is the degree of the Gauss map $\frac{v}{||v||}$ from $\mathbb{S}_\varepsilon$ into the unit sphere in $\mathbb{R}^{m'}$.

If v and v' are two such vector fields, then their local indices at a coincide if and only if their Gauss maps are homotopic (special case of Hopf Theorem [120]). That is equivalent to saying that their restrictions to the sphere $\mathbb{S}_\varepsilon$ are homotopic.

Let us consider now an m'-dimensional smooth manifold M, then a *vector field* on M is a section of its tangent bundle TM. Giving a local chart $(x_1, \dots, x_{m'})$ on M, a vector field on M is locally expressed as above and the definition of the local index at an isolated singularity extends in the obvious way. The index does not depend on the local chart.

Definition 1.1.1. The *total index* of v, denoted

$$\mathrm{Ind}_{\mathrm{PH}}(v, M),$$

is the sum of all its local indices at the singular points.

A fundamental property of the total index is the following classical theorem:

Theorem 1.1.1. (Poincaré–Hopf) *Let M be a closed, oriented manifold and v a continuous vector field on M with finitely many isolated singularities. Then one has*

$$\mathrm{Ind}_{\mathrm{PH}}(v, M) = \chi(M),$$

independently of v, where $\chi(M)$ denotes the Euler–Poincaré characteristic of M.

If M is now an oriented manifold with boundary, one has a similar theorem:

Theorem 1.1.2. *Let M be a compact, oriented m'-manifold with boundary ∂M, and let v be a nonsingular vector field on a neighborhood U of ∂M. Then:*

(1) *v can be extended to the interior of M with finitely many isolated singularities.*
(2) *The total index of v in M is independent of the way we extend it to the interior of M. In other words, the total index of v is fully determined by its behavior near the boundary.*
(3) *If v is everywhere transverse to the boundary and pointing outwards from M, then one has $\mathrm{Ind}_{\mathrm{PH}}(v, M) = \chi(M)$. If v is everywhere transverse to ∂M and pointing inwards M, then $\mathrm{Ind}_{\mathrm{PH}}(v, M) = \chi(M) - \chi(\partial M)$.*

Remark 1.1.1. It is worth saying that although $\mathrm{Ind}_{\mathrm{PH}}(v, M)$ is determined by its behavior near the boundary, it does depend on the topology of the interior of M. In fact a formula of Morse and Pugh (c.f. [124, 133, 145]) provides an explicit way to compute the index out of boundary data, generalizing a classical formula of Poincaré for vector fields on the plane.

We remark also that one of the basic properties of the index is its stability under perturbations. In other words, if v has an isolated singularity at a point a in a manifold M of index $\mathrm{Ind}(v, a)$ and we make a small perturbation of v to get a new vector field $\hat{v}$ with isolated singularities, then $\mathrm{Ind}(v, a)$ will be the sum of the local indices of $\hat{v}$ at its singular points near a. In fact it is well-known that every vector field can be *morsified*, *i.e.*, approximated by vector fields whose singularities are nondegenerate. Each such singularity has local index ± 1 and the number of such points, counted with signs, equals the index of v at a. In short, the local index of v at a is the number of singularities, counted with sign, into which a splits under a morsification of v. We will see later that this basic property has its analogues in the case of vector fields on singular varieties.

This stability of the index is also preserved for vector fields with nonisolated singularities. To make this precise we need to introduce a few concepts, which will also be used later.

The following property of the local index is well-known and we leave the proof as an exercise:

Proposition 1.1.1. *Let v be a vector field around $0 \in \mathbb{R}^{m'}$ with an isolated singularity at 0 of index $\mathrm{Ind}(v, 0)$, and let w be a vector field around $0 \in \mathbb{R}^{n'}$ with an isolated singularity at 0 of index $\mathrm{Ind}(w, 0)$. Then the direct product $v \oplus w$ is a vector field in $\mathbb{R}^{m'+n'}$ with an isolated singularity of index $\mathrm{Ind}(v, 0) \cdot \mathrm{Ind}(w, 0)$.*

A consequence of this result is the well-known fact that if M, N are closed, oriented manifolds, then $\chi(M \times N) = \chi(M) \cdot \chi(N)$. Another consequence of 1.1.1 that will be used later is that if we have a vector field v in $\mathbb{R}^{m'}$ with an isolated singularity at 0 of index $\mathrm{Ind}(v, 0)$, and if we extend it to $\mathbb{R}^{m'} \times \mathbb{R}^{n'}$ by taking the vector field $w = \sum_{i=1}^{n'} y_i \partial/\partial y_i$ in $\mathbb{R}^{n'}$, then the index does not change, where $(y_1, \ldots, y_{n'})$ are the coordinates on $\mathbb{R}^{n'}$. If we took the vector field $-\sum_{i=1}^{r} y_i \partial/\partial y_i + \sum_{i=r+1}^{n'} y_i \partial/\partial y_i$ in $\mathbb{R}^{n'}$, then the index in $\mathbb{R}^{m'} \times \mathbb{R}^{n'}$ would be $\pm\mathrm{Ind}(v, 0)$, depending on the parity of the number r of negative signs.

1.1.2 Poincaré–Hopf Index at Nonisolated Points

In the following, singularities of the vector field v are not necessarily isolated points. We still define a Poincaré–Hopf index in that case.

Let M be a manifold with boundary ∂M. Let us consider a triangulation (K) of M compatible with the boundary and (K') a barycentric subdivision of (K). Using (K') one constructs the associated cellular dual decomposition (D) of M: given a simplex σ in (K) of dimension s, its *dual* $d(\sigma)$ is the union of all simplices τ in (K') whose closure meets σ exactly at its barycenter $\hat{\sigma}$, that is $\overline{\tau} \cap \sigma = \hat{\sigma}$. If σ is in the interior of M, that is a cell, if σ is in the boundary of M, that is a "half-cell." It is an exercise to see that the dimension of $d(\sigma)$ is $m' - s$. Taking the union of all these dual cells (or half-cells) we get the dual decomposition (D) of (K); by construction its cells and half-cells are all transverse to (K) (we refer to [25] for details including orientation notions).

Let S be a compact connected (K)-subcomplex of the interior of M.

Definition 1.1.2. A *cellular tube* $\mathcal{T}$ around S in M is the union of cells (D) which are dual of simplices in S.

This notion generalizes the concept of tubular neighborhood of a submanifold S. If S is a submanifold without boundary, then $\mathcal{T}$ is a bundle on S, whose fibers are discs. In general, that is not the case.

Remark 1.1.2. A cellular tube $\mathcal{T}$ around S has the following properties :
(1) $\mathcal{T}$ is a compact neighborhood of S, containing S in its interior and $\partial\mathcal{T}$ is a retract of $\mathcal{T} \setminus S$.
(2) $\mathcal{T}$ is a *regular* neighborhood of S, thus $\mathcal{T}$ retracts to S.
(3) We can assume the cellular tubes in M have smooth boundary [83].

Let us denote by U a neighborhood of S in M. If the triangulation is sufficiently "fine," then we can assume $\mathcal{T} \subset U$.

According to Theorem 1.1.2, a nonsingular continuous vector field v on a neighborhood of $\partial\mathcal{T}$ can be extended to the interior of $\mathcal{T}$ with finitely many isolated singularities. The total index of v on $\mathcal{T}$ is defined as the sum of the indices of the extension of v at these points.

Definition 1.1.3. Let v be a continuous vector field on a neighborhood U of S in M, nonsingular on $U \setminus S$, then the *Poincaré–Hopf index* of v at S, denoted $\mathrm{Ind}_{\mathrm{PH}}(v, S)$ (or simply by $\mathrm{Ind}(v, S)$, if there is no ambiguity), is defined as $\mathrm{Ind}_{\mathrm{PH}}(v, \mathcal{T})$.

This number $\mathrm{Ind}_{\mathrm{PH}}(v, S)$ depends only on the behavior of v near S and not on the choice of the neighborhood U, or of the tube $\mathcal{T}$. Moreover, for this index it does not matter what actually happens on S, we only care what happens around S, but away from S. In particular, if v is "radial" from S, *i.e.*, if it is transverse to the boundary of a cellular tube around S pointing outward, then $\mathrm{Ind}_{\mathrm{PH}}(v, S) = \chi(S)$.

Now let M be a compact oriented C^∞ manifold possibly with boundary ∂M and v a continuous vector field on M, nonsingular on the boundary. From the above considerations, we may assume that the set $S(v)$ of singular points of v has only a finite number of components $\{S_\lambda\}$.

If M has no boundary, the Poincaré–Hopf Theorem implies that

$$\sum_\lambda \mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda) = \chi(M). \tag{1.1.3}$$

If M has a boundary, the sum $\sum_\lambda \mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda)$ depends only on the behavior of v near ∂M. For example, if v is pointing outwards everywhere on ∂M, then we have the same formula (1.1.3). If v is pointing inwards everywhere on ∂M, the right hand side becomes $\chi(M)-\chi(\partial M)$. In particular, if the (real) dimension of M is even (as it will usually be the case in this book) and if v is everywhere transverse to ∂M, then we have again the same formula (1.1.3).

Here we introduce the concept of *the difference* which will be used in the rest of the book. For this we let v and v' be continuous vector fields on a neighborhood U of S in M, nonsingular on $U \setminus S$. Let $\mathcal{T}$ and $\mathcal{T}'$ be cellular tubes around S in U such that interior of $\mathcal{T}$ contains the closure of $\mathcal{T}'$ and denote $X = \mathcal{T} \setminus \mathcal{T}'$. Let us consider w a vector field on X with isolated singularities which restricts to v on $\partial\mathcal{T}$ and to v' on $\partial\mathcal{T}'$; such a vector field w always exists by Theorem 1.1.2. We may denote by $d(v, v') = \mathrm{Ind}_{\mathrm{PH}}(w, X)$ *the difference* between v and v'. Then one has:

$$\mathrm{Ind}_{\mathrm{PH}}(v, S) = \mathrm{Ind}_{\mathrm{PH}}(v', S) + d(v, v'). \tag{1.1.4}$$

One can easily prove the following result that will be used later.

Proposition 1.1.2. *Let M_1 and M_2 be compact oriented m'-manifolds, $m' > 1$, with the same boundary $N = \partial M_1 = \partial M_2$, and let v be a non-singular vector field defined on a neighborhood of N. Then one has:*

$$\mathrm{Ind}_{\mathrm{PH}}(v, M_1) - \mathrm{Ind}_{\mathrm{PH}}(v, M_2) = \chi(M_1) - \chi(M_2).$$

1.2 Poincaré and Alexander Dualities

We briefly review the classical case, which will be generalized to the case of singular varieties in Sect. 10.4 below. In either case, we follow the descriptions given in [25].

Let M be an oriented manifold of real dimension m'. We take a triangulation (K) of M and the cellular decomposition (D) dual to (K), as before. The groups of chains relative to (K) and (D) are denoted by $C_*^{(K)}(M)$ and $C_*^{(D)}(M)$, respectively. Also, the groups of cochains relative to (K) and (D) are denoted by $C^*_{(K)}(M)$ and $C^*_{(D)}(M)$, respectively. The intersection of an

i-simplex σ and its dual $(m'-i)$-cell $d(\sigma)$ is transverse and consists of one point, the barycenter $\hat{\sigma}$ of σ.

First, if M is compact, we define a homomorphism

$$P : C^{m'-i}_{(D)}(M) \longrightarrow C^{(K)}_i(M) \qquad \text{by} \qquad P(c) = \sum_{\sigma} \langle c, d(\sigma) \rangle \, \sigma \tag{1.2.1}$$

for an $(m'-i)$-cochain c, where the sum is taken over all i-simplices σ of M (we follow the orientation conventions in [25]). This induces the Poincaré isomorphism

$$P_M : H^{m'-i}(M) \xrightarrow{\sim} H_i(M).$$

Next, let S be a (K)-subcomplex of M whose geometric realization is also denoted by S. Let $C^*_{(D)}(M, M \setminus S)$ denote the subgroup of $C^*_{(D)}(M)$ consisting of cochains which are zero on the cells not intersecting with S.

Suppose S is compact (M may not be compact). Then we may define a homomorphism

$$A : C^{m'-i}_{(D)}(M, M \setminus S) \longrightarrow C^{(K)}_i(S)$$

taking, in the sum in (1.2.1), only i-simplices of S. This induces the Alexander isomorphism

$$A_{M,S} : H^{m'-i}(M, M \setminus S) \xrightarrow{\sim} H_i(S).$$

From the construction, we have the following

Proposition 1.2.1. *If M is compact, we have the commutative diagram*

$$\begin{array}{ccc} H^{m'-i}(M, M \setminus S) & \xrightarrow{j^*} & H^{m'-i}(M) \\ \wr \downarrow A_{M,S} & & \wr \downarrow P_M \\ H_i(S) & \xrightarrow{i_*} & H_i(M). \end{array}$$

1.3 Chern Classes via Obstruction Theory

1.3.1 Chern Classes of Almost Complex Manifolds

Let us recall the definition of the Chern classes via obstruction theory [28, 89, 123, 153]. This can be done in full generality, however for simplicity we consider first the case of Chern classes of almost-complex manifolds, and later in this section we indicate how this generalizes to complex vector bundles in general.

Now we assume we are given an almost complex $m' = 2m$-manifold M, so its tangent bundle TM is endowed with the structure of a complex vector bundle of rank m.

Definition 1.3.1. An *r-field* on a subset A of M is a set $v^{(r)} = \{v_1, \dots, v_r\}$ of r continuous vector fields defined on A. A singular point of $v^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A nonsingular r-field is also called an *r-frame.*

Let $W_{r,m}$ be the Stiefel manifold of complex r-frames in $\mathbb{C}^m$. Notice that we will use r-frames which are not necessarily orthonormal, but this does not change the results, because every frame is homotopic to an orthonormal one. We know (see [153]) that $W_{r,m}$ is $(2m - 2r)$-connected and its first nonzero homotopy group is $\pi_{2m-2r+1}(W_{r,m}) \simeq \mathbb{Z}$. The bundle of r-frames on M, denoted by $W_r(TM)$, is the bundle associated with the tangent bundle and whose fiber over $x \in M$ is the set of r-frames in T_xM (diffeomorphic to $W_{r,m}$). In the following, we fix the notation $q = m - r + 1$.

The Chern class $c^q(M) \in H^{2q}(M)$ is the first possibly nonzero obstruction to constructing a section of $W_r(TM)$. Let us recall the standard obstruction theory process to construct this class. Let σ be a k-cell of the given cellular decomposition (D), contained in an open subset $U \subset M$ on which the bundle $W_r(TM)$ is trivialized. If the section $v^{(r)}$ of $W_r(TM)$ is already defined over the boundary of σ, it defines a map:

$$\partial\sigma \simeq \mathbb{S}^{k-1} \xrightarrow{v^{(r)}} W_r(TM)|_U \simeq U \times W_{r,m} \xrightarrow{pr_2} W_{r,m},$$

thus an element of $\pi_{k-1}(W_{r,m})$.

If $k \leq 2m - 2r + 1$, this homotopy group is zero, so the section $v^{(r)}$ can be extended to σ without singularity. It means that we can always construct a section $v^{(r)}$ of $W_r(TM)$ over the $(2q - 1)$-skeleton of (D).

If $k = 2(m - r + 1) = 2q$, we meet an obstruction. The r-frame on the boundary of each cell σ defines an element, denoted by $\mathrm{Ind}(v^{(r)}, \sigma)$, in the homotopy group $\pi_{2q-1}(W_{r,m}) \simeq \mathbb{Z}$.

Definition 1.3.2. The integer $\mathrm{Ind}(v^{(r)}, \sigma)$ is *the (Poincaré–Hopf) index* of the r-frame $v^{(r)}$ on the cell σ.

Notice that for this index, to be well defined, we need that the cell σ has the correct dimension. This will be essential for our considerations in Chap. 10.

The generators of $\pi_{2q-1}(W_{r,m})$ being consistent (see [153]), this defines a cochain

$$\gamma \in C^{2q}(M; \pi_{2q-1}(W_{r,m})),$$

by setting $\gamma(\sigma) = \mathrm{Ind}(v^{(r)}, \sigma)$, for each $2q$-cell σ, and then by extending it linearly. This cochain is actually a cocycle and the cohomology class that it represents is the q-th *Chern class* $c^q(M)$ of M in $H^{2q}(M)$.

The class one gets in this way is independent of the various choices involved in its definition. Note that $c^m(M)$ coincides with the Euler class of the underlying real tangent bundle $T_{\mathbb{R}}M$, so these classes are natural generalization of the Euler class.

There is another useful definition of the index $\mathrm{Ind}(v^{(r)}, \sigma)$: let us write the frame $v^{(r)}$ as $(v^{(r-1)}, v_r)$, where the last vector is individualized, and suppose that $v^{(r)}$ is already defined on $\partial\sigma$. There is no obstruction to extending the $(r-1)$-frame $v^{(r-1)}$ from $\partial\sigma$ to σ because the dimension of the obstruction for such an extension is $2(m-(r-1)+1) = \dim\sigma + 2$. The $(r-1)$-frame $v^{(r-1)}$, defined on σ, generates a complex subbundle G^{r-1} of rank $(r-1)$ of $TM|_\sigma$ and one can write

$$TM|_\sigma \simeq G^{r-1} \oplus Q^q \,,$$

where Q^q is an orthogonal complement of (complex) rank $q = m-(r-1)$.

The obstruction to extending the last vector v_r inside a $2q$-simplex σ as a nonvanishing section of Q^q is given by an element of $\pi_{2q-1}(\mathbb{C}^q \setminus \{0\}) \simeq \mathbb{Z}$ corresponding to the composition of the map $v_r : \partial\sigma \simeq \mathbb{S}^{2q-1} \longrightarrow Q^q|_U$ with the projection on the fiber $\mathbb{C}^q \setminus \{0\}$. Let us denote by $\mathrm{Ind}_{Q^q}(v_r, \sigma)$ the integer so obtained. The obstruction to extending the r-frame $v^{(r)}|_{\partial\sigma}$ inside σ as an r-frame tangent to M is the same as the obstruction to extending the last vector v_r inside σ as a non zero section of Q^q. In fact there is a natural isomorphism $\pi_{2q-1}(W_{r,m}) \simeq \pi_{2q-1}(\mathbb{C}^q \setminus \{0\})$ (for compatible orientations) and by this isomorphism we have the equality of integers

$$\mathrm{Ind}(v^{(r)}, \sigma) = \mathrm{Ind}_{Q^q}(v_r, \sigma).$$

A different choice of $v^{(r-1)}$ gives another choices of v_r and of Q^q, however all such bundles Q^q are homotopic and the index we obtain is the same.

Remark 1.3.1. The Chern classes of complex vector bundles in general are defined in essentially the same way as above. If E is a complex vector bundle of rank $k > 0$ over a locally finite simplicial complex B of dimension $n \geq k$, then one has *Chern classes* $c^i(E) \in H^{2i}(B;\mathbb{Z}), i = 1, \ldots, k$. The class $c^i(E)$ is by definition the primary obstruction to constructing $(k-i+1)$ linearly independent sections of E.

The class $c^0(E)$ is defined to be 1 and one has *the total Chern class* of E defined by:

$$c^*(E) = 1 + c^1(E) + \cdots + c^k(E)$$

This can be regarded as an element in the cohomology ring $H^*(B)$ and it is invertible in this ring.

1.3.2 Relative Chern Classes

Suppose now that (L) is a sub-complex of (D) whose geometric realization $|L|$ is also denoted by L. Assume that we are already given an r-frame $v^{(r)}$ on the $2q$-skeleton of L, denoted by $L^{(2q)}$. The same arguments as before say that we can always extend $v^{(r)}$ without singularity to $L^{(2q)} \cup D^{(2q-1)}$. If we wish to extend this frame to the $2q$-skeleton of (D) we meet an obstruction for each corresponding cell which is not in (L). This gives rise to a cochain which vanishes on L and is a cocycle in $H^{2q}(M, L)$.

Definition 1.3.3. The *relative Chern class*

$$c^q(M, L; v^{(r)}) \in H^{2q}(M, L),$$

is the class represented by the previous cocycle.

The image of $c^q(M, L; v^{(r)})$ by the natural map in $H^{2q}(M)$ is the usual Chern class but as a relative class it does depend on the choice of the frame $v^{(r)}$ on L. Let us discuss how the relative Chern class varies as we change the r-frame.

If we have two frames $v_1^{(r)}$ and $v_2^{(r)}$ on $L^{(2q)}$ the difference between the corresponding classes is given by the difference cocycle of the frames on L; in the product $L \times I$, suppose $v_1^{(r)}$ is defined at the level $L \times \{0\}$ and $v_2^{(r)}$ is defined at the level $L \times \{1\}$, then the difference cocycle $d(v_1^{(r)}, v_2^{(r)})$ is well defined in

$$H^{2q}(L \times I, L \times \{0\} \cup L \times \{1\}) \simeq H^{2q-1}(L),$$

as the obstruction to the extension of the given sections on the boundary of $L \times I$ ([153] Sect. 33.3). As shown in [153], we have the following formula:

$$c^q(M, L; v_2^{(r)}) = c^q(M, L; v_1^{(r)}) + \delta d(v_1^{(r)}, v_2^{(r)}),$$

where $\delta : H^{2q-1}(L) \to H^{2q}(M, L)$ is the connecting homomorphism. Also, for three frames $v_1^{(r)}$, $v_2^{(r)}$, and $v_3^{(r)}$ as above, we have

$$d(v_1^{(r)}, v_3^{(r)}) = d(v_1^{(r)}, v_2^{(r)}) + d(v_2^{(r)}, v_3^{(r)}) \qquad (1.3.1)$$

For $r = 1$ the frames consist of a single vector field and the difference above corresponds, via Poincaré duality, to the one previously defined for vector fields (cf. 1.1.4).

In the sequel, we will show that the relative Chern class allows us to define Chern class in homology.

Let S be a compact (K)-subcomplex of M, and U a neighborhood of S. Let $\mathcal{T}$ be a cellular tube in U around S. Take an r-field $v^{(r)}$ defined on $D^{(2q)}$, possibly with singularities. We suppose that the only singularities inside U are located in S. This implies that $v^{(r)}$ has no singularities on $(\partial\mathcal{T})^{(2q)}$ so there is a well defined relative Chern class (see 1.3.3)

$$c^q(\mathcal{T}, \partial\mathcal{T}; v^{(r)}) \in H^{2q}(\mathcal{T}, \partial\mathcal{T}).$$

Definition 1.3.4. The *Poincaré–Hopf class* of $v^{(r)}$ at S, which is denoted by $\mathrm{PH}(v^{(r)}, S)$, is the image of $c^q(\mathcal{T}, \partial\mathcal{T}; v^{(r)})$ by the isomorphism $H^{2q}(\mathcal{T}, \partial\mathcal{T}) \simeq H^{2q}(\mathcal{T}, \mathcal{T} \setminus S)$ followed by the Alexander duality (see [25])

$$A_M : H^{2q}(\mathcal{T}, \mathcal{T} \setminus S) \xrightarrow{\sim} H_{2r-2}(S). \tag{1.3.2}$$

For $r = 1$ the frame consists of a single vector field v and the class $\mathrm{PH}(v, S) \in H_0(S)$ is identified with the Poincaré–Hopf index of v at S, $\mathrm{Ind}_{\mathrm{PH}}(v, S)$, previously defined (Definition 1.1.3).

Note that if $\dim S < 2r - 2$, then $\mathrm{PH}(v^{(r)}, S) = 0$.

The relation between the Poincaré–Hopf class of $v^{(r)}$ and the index we defined above is the following:

$$\mathrm{PH}(v^{(r)}, S) = \sum \mathrm{Ind}(v^{(r)}, d(\sigma))\, \sigma\, ,$$

where the sum runs over the $2(r-1)$-simplices σ of the triangulation of S and $d(\sigma)$ is the dual cell of σ (of dimension $2q$).

Let us consider now the case of manifolds with boundary. Let M be a compact almost complex $2m$-manifold, with nonempty boundary ∂M. Let (K) be a triangulation of M compatible with ∂M. The union of all "half-cells" dual to simplices in ∂M, denoted by $\mathcal{U}$ is a regular neighborhood of ∂M. Its boundary is denoted by $\partial\mathcal{U}$, which is a union of (D)-cells and is homeomorphic to ∂M. The pair $(M \setminus (\mathrm{Int}\, \mathcal{U}), \partial\mathcal{U})$ is homeomorphic to $(M, \partial M)$ and one can apply the previous construction.

Let $v^{(r)}$ be an r-field on the $(2q)$-skeleton of (D), with singularities located on a compact subcomplex S in $M \setminus (\mathrm{Int}\, \mathcal{U})$. On the $(2q)$-skeleton of $\mathcal{U}$, we have a well defined r-frame $v^{(r)}$. Let $\{S_\lambda\}$ be the connected components of S. Then, by setting $c_{r-1}(M; v^{(r)}) = c^q(M, \partial M; v^{(r)}) \frown [M, \partial M]$, we have

$$\sum_\lambda (i_\lambda)_* \mathrm{PH}(v^{(r)}, S_\lambda) = c_{r-1}(M; v^{(r)}) \qquad \text{in } \ H_{2r-2}(M), \tag{1.3.3}$$

where $i_\lambda : S_\lambda \hookrightarrow M$ is the inclusion.

In particular, the sum of the Poincaré–Hopf classes is determined by the behavior of $v^{(r)}$ near ∂M and does not depend on the extension to the interior of M. Note that we may assume that $v^{(r)}$ is nonsingular on $D^{(2q-1)}$.

If $r = 1$ and $v^{(1)} = \{v\}$, the relative Chern class is also called the *Euler class of M relative to v* and its evaluation on the orientation cycle of $(M, \partial M)$ gives the index of v on M. Thus, if v is everywhere transverse to the boundary, the formula (1.3.3) reduces to (1.1.3).

Remark 1.3.2. In the sequel we often speak of *localizing* Chern classes, which can be done by two different methods: either obstruction theory or Chern–Weil theory. The obstruction theoretical viewpoint comes from the above concept of relative Chern classes: if S is a compact sub-complex of M, U a tubular neighborhood of S, and we are given an r-frame on the intersection with $U \setminus S$ with the appropriate skeleton (for some triangulation or cellular decomposition of M), then the cycle that represents the corresponding Chern class c^q vanishes on ∂U. Hence we have a contribution for c^q that is localized in S, and another contribution in the complement of U. In the following sections the geometric counterpart for making these localizations will be to consider connections that are flat in the linear subspaces determined by the frame. If $r = 1$ and S is a point, the "localization" one gets is simply the contribution to $\chi(M)$ given by the local Poincaré–Hopf index of a vector field at the isolated singularity.

1.4 Chern–Weil Theory of Characteristic Classes

In this section, we briefly review how to define characteristic classes of complex vector bundles using connections. This approach allows us to obtain precise results. If we combine this with the Čech-de Rham cohomology, this method is particularly effective when we deal with the "localization problem."

Let M be a C^∞ manifold of dimension m'. For an open set U in M, we denote by $A^p(U)$ the complex vector space of complex valued C^∞ p-forms on U. Also, for a C^∞ complex vector bundle E of rank k on M, we let $A^p(U, E)$ be the vector space of "E-valued p-forms" on U, *i.e.*, C^∞ sections of the bundle $\bigwedge^p (T^c_{\mathbb{R}} M)^* \otimes E$ on U, where $(T^c_{\mathbb{R}} M)^*$ denotes the dual of the complexification of the real tangent bundle $T_{\mathbb{R}} M$ of M. Thus $A^0(U)$ is the ring of C^∞ functions and $A^0(U, E)$ is the $A^0(U)$-module of C^∞ sections of E on U.

Definition 1.4.1. A *connection* for E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M, E) \longrightarrow A^1(M, E)$$

satisfying the "Leibniz rule":

$$\nabla(fs) = df \otimes s + f\nabla(s) \quad \text{for} \quad f \in A^0(M) \text{ and } s \in A^0(M, E).$$

Example 1.4.1. The exterior derivative

$$d : A^0(M) \longrightarrow A^1(M)$$

is a connection for the trivial line bundle $M \times \mathbb{C}$.

From the definition we have the following:

Lemma 1.4.1. *A connection ∇ is a local operator, i.e., if a section s is identically 0 on an open set U, so is $\nabla(s)$.*

Thus the restriction of ∇ to an open set U makes sense and it is a connection for $E|_U$.

Definition 1.4.2. Let ∇ be a connection for E on U. For a nonvanishing section s of E on U, we say that ∇ is s-*trivial*, if $\nabla(s) = 0$. More generally, for an r-frame $\mathbf{s} = (s_1, \dots, s_r)$, ∇ is $\mathbf{s}$-*trivial*, if $\nabla(s_i) = 0,\ i = 1, \dots, r$.

Thus in Example 1.4.1, ∇ is trivial with respect to an arbitrary (nonzero) constant section. From the definition we also have the following lemma.

Lemma 1.4.2. *Let $\nabla_1, \dots, \nabla_\ell$ be connections for E and $f_1, \dots, f_\ell$ C^∞ functions on M with $\sum_{i=1}^{\ell} f_i \equiv 1$. Then $\sum_{i=1}^{\ell} f_i \nabla_i$ is a connection for E.*

One of the consequences of the above lemmas is that every vector bundle admits a connection. This can be shown by taking an open covering $\mathcal{U}$ of M so that E is trivial on each open set in $\mathcal{U}$, choosing a connection on each open set trivial with respect to some frame of E, and then patching them together by a partition of unity subordinate to $\mathcal{U}$.

If ∇ is a connection for E, it induces a $\mathbb{C}$-linear map

$$\nabla : A^1(M, E) \longrightarrow A^2(M, E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s) \quad \text{for} \quad \omega \in A^1(M) \text{ and } s \in A^0(M, E).$$

The composition

$$K = \nabla \circ \nabla : A^0(M, E) \longrightarrow A^2(M, E)$$

is called the *curvature* of ∇. It is not difficult to see that

$$K(fs) = fK(s) \quad \text{for} \quad f \in A^0(M) \text{ and } s \in A^0(M, E).$$

The fact that a connection is a local operator allows us to obtain local representations of it and its curvature by matrices whose entries are differential forms. Thus suppose that ∇ is a connection for a vector bundle E of rank k and that E is trivial on U. If $\mathbf{e} = (e_1, \dots, e_k)$ is a frame of E on U, we may write, for $i = 1, \dots, k$,

$$\nabla(e_i) = \sum_{j=1}^{k} \theta_{ji} \otimes e_j, \quad \theta_{ji} \in A^1(U).$$

We call $\theta = (\theta_{ij})$, the matrix whose (i, j) entry is θ_{ij}, the connection matrix of ∇ with respect to $\mathbf{e}$. For an arbitrary section s on U, we may write $s = \sum_{i=1}^{k} f_i e_i$ where the f_i are C^∞ functions on U and we compute

$$\nabla(s) = \sum_{i=1}^{k}(df_i + \sum_{j=1}^{k}\theta_{ij}f_j) \otimes e_i.$$

Note that the connection ∇ is $\mathbf{e}$-trivial if and only if $\theta = 0$. Thus in this case we have $\nabla(s) = \sum_{i=1}^{k} df_i \otimes e_i$. Also, from the definition we get

$$K(e_i) = \sum_{j=1}^{k} \kappa_{ji} \otimes e_j, \qquad \kappa_{ij} = d\theta_{ij} + \sum_{\ell=1}^{k} \theta_{i\ell} \wedge \theta_{\ell j}.$$

We call $\kappa = (\kappa_{ij})$ the curvature matrix of ∇ with respect to $\mathbf{e}$. If $\mathbf{e}' = (e'_1 \ldots, e'_k)$ is another frame of E on U', we have $e'_i = \sum_{j=1}^{k} a_{ji}e_j$ for some C^∞ functions a_{ji} on $U \cap U'$. The matrix $A = (a_{ij})$ is nonsingular at each point of $U \cap U'$. If we denote by θ' and κ' the connection and curvature matrices of ∇ with respect to $\mathbf{e}'$,

$$\theta' = A^{-1} \cdot dA + A^{-1}\theta A \quad \text{and} \quad \kappa' = A^{-1}\kappa A \quad \text{in} \quad U \cap U'. \tag{1.4.1}$$

Let $m = [m'/2]$ and, for each $i = 1, \ldots, m$, let σ_i denote the ith elementary symmetric function in m variables $X_1, \ldots, X_m$, *i.e.*, $\sigma_i(X_1, \ldots, X_m)$ is a polynomial of degree i defined by

$$\prod_{i=1}^{m}(1 + X_i) = 1 + \sigma_1(X_1, \ldots, X_m) + \cdots + \sigma_m(X_1, \ldots, X_m).$$

Since differential forms of even degrees commute with one another with respect to the exterior product, we may treat κ as an ordinary matrix whose entries are numbers. We define a $2i$-form $\sigma_i(\kappa)$ on U by

$$\det(I + \kappa) = 1 + \sigma_1(\kappa) + \cdots + \sigma_m(\kappa),$$

where I denotes the identity matrix of rank k. Note that $\sigma_i(\kappa) = 0$ for $i = k+1, \ldots, m$, and in particular, $\sigma_1(\kappa)$ is the trace $\operatorname{tr}(\kappa)$ and $\sigma_k(\kappa)$ is the determinant $\det(\kappa)$. Although $\sigma_i(\kappa)$ depends on the connection ∇, by (1.4.1), it does not depend on the choice of the frame of E and it defines a global $2i$-form on M, which we denote by $\sigma_i(\nabla)$. It is shown that the form is closed ([75, Ch.3, 3 Lemma], [123, Appendix C, Fundamental Lemma]). We set

$$c^i(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sigma_i(\nabla)$$

and call it the i-th Chern form.

If we have two connections ∇ and ∇' for E, there is a $(2i-1)$-form $c^i(\nabla, \nabla')$ satisfying

$$c^i(\nabla, \nabla') = -c^i(\nabla', \nabla) \qquad \text{and} \qquad d\,c^i(\nabla, \nabla') = c^i(\nabla') - c^i(\nabla). \quad (1.4.2)$$

In fact the form $c^i(\nabla, \nabla')$ is constructed as follows. We consider the vector bundle $E \times \mathbb{R} \to M \times \mathbb{R}$ and define the connection $\widetilde{\nabla}$ for it by

$$\widetilde{\nabla} = (1-t)\nabla + t\nabla',$$

where t denotes a coordinate on $\mathbb{R}$. Denoting by $[0,1]$ the unit interval and by $\pi : M \times [0,1] \to M$ the projection, we have the integration along the fiber

$$\pi_* : A^{2i}(M \times [0,1]) \longrightarrow A^{2i-1}(M).$$

Then we set

$$c^i(\nabla, \nabla') = \pi_*(c^i(\widetilde{\nabla})). \quad (1.4.3)$$

A similar construction works for an arbitrary collection of finite number of connections and the resulting differential form is called the *Bott difference form* ([19, p. 65]).

From the above, we see that the class $[c^i(\nabla)]$ of the closed $2i$-form $c^i(\nabla)$ in the de Rham cohomology $H^{2i}(M, \mathbb{C})$ depends only on E and not on the choice of the connection ∇. We denote this class by $c^i(E)$ and call it the i-th Chern class $c^i(E)$ of E via the Chern–Weil theory. We call

$$c(E) = 1 + c^1(E) + \cdots + c^k(E)$$

the total Chern class of E, which is considered as an element in the cohomology ring $H^*(M, \mathbb{C})$. Note that the class $c(E)$ is invertible in $H^*(M, \mathbb{C})$.

Remark 1.4.1. 1. It is known (see, e.g., [123]) that the class $c^i(E)$ defined as above is the image of the class $c^i(E)$ in $H^{2i}(M, \mathbb{Z})$ defined via the obstruction theory by the canonical homomorphism

$$H^{2i}(M, \mathbb{Z}) \longrightarrow H^{2i}(M, \mathbb{C}).$$

This fact can also be proved directly using an expression of the mapping degree in terms of connections (see, e.g., [161]).

2. Let H be a hyperplane in the projective space $\mathbb{CP}^m$. For the hyperplane bundle L_H, the line bundle determined by H, we have

$$c(L_H) = 1 + h_m,$$

where h_m denotes the canonical generator of $H^2(\mathbb{CP}^m, \mathbb{C})$ (the Poincaré dual of the homology class $[\mathbb{CP}^{m-1}]$). See Sect. 1.6.4 for the proof of a more precise statement.

More generally, if we have a symmetric polynomial φ, we may write $\varphi = P(\sigma_1, \sigma_2, \dots)$ for some polynomial P. We define, for a connection ∇ for E, the characteristic form $\varphi(\nabla)$ for φ by $\varphi(\nabla) = P(c^1(\nabla), c^2(\nabla), \dots)$, which is a closed form and defines the characteristic class $\varphi(E)$ of E for φ in the de Rham cohomology. We may also define the difference form $\varphi(\nabla, \nabla')$ by a construction similar to the one for the Chern polynomials.

1.5 Čech-de Rham Cohomology

In the subsequent sections, we discuss "localizations of characteristic classes" and for this purpose, the Chern–Weil theory adapted to the Čech-de Rham cohomology is particularly relevant. The Čech-de Rham cohomology is defined for an arbitrary covering of a manifold M, however for simplicity here we only consider coverings of M consisting of two open sets. For details, we refer to [20] and [156].

Let M be a C^∞ manifold of dimension m' and $\mathcal{U} = \{U_0, U_1\}$ an open covering of M. We set $U_{01} = U_0 \cap U_1$. Define the vector space $A^p(\mathcal{U})$ as

$$A^p(\mathcal{U}) = A^p(U_0) \oplus A^p(U_1) \oplus A^{p-1}(U_{01}).$$

Thus an element ξ in $A^p(\mathcal{U})$ is given by a triple $\xi = (\xi_0, \xi_1, \xi_{01})$ with ξ_0 a p-form on U_0, ξ_1 a p-form on U_1 and ξ_{01} a $(p-1)$-form on U_{01}.

We define the operator $D : A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U})$ by

$$D\xi = (d\xi_0, d\xi_1, \xi_1 - \xi_0 - d\xi_{01}).$$

Then it is not difficult to see that $D \circ D = 0$. This allows us to define a cohomological complex, *the Čech-de Rham complex* :

$$\cdots \longrightarrow A^{p-1}(\mathcal{U}) \overset{D^{(p-1)}}{\longrightarrow} A^p(\mathcal{U}) \overset{D^{(p)}}{\longrightarrow} A^{p+1}(\mathcal{U}) \longrightarrow \cdots$$

Set $Z^p(\mathcal{U}) = \mathrm{Ker} D^p$, $B^p(\mathcal{U}) = \mathrm{Im} D^{p-1}$ and

$$H^p_D(\mathcal{U}) = Z^p(\mathcal{U})/B^p(\mathcal{U}),$$

which is called the p-th Čech-de Rham cohomology of $\mathcal{U}$. We denote the image of ξ by the canonical surjection $Z^p(\mathcal{U}) \to H^p_D(\mathcal{U})$ by $[\xi]$.

Theorem 1.5.1. *The map $A^p(M) \to A^p(\mathcal{U})$ given by $\omega \mapsto (\omega, \omega, 0)$ induces an isomorphism*

$$\alpha : H^p_{dR}(M) \xrightarrow{\sim} H^p_D(\mathcal{U}).$$

Proof. It is not difficult to show that α is well-defined. To prove that α is surjective, let $\xi = (\xi_0, \xi_1, \xi_{01})$ be such that $D\xi = 0$. Let $\{\rho_0, \rho_1\}$ be a partition of unity subordinated to the covering $\mathcal{U}$. Define $\omega = \rho_0\xi_0 + \rho_1\xi_1 - d\rho_0 \wedge \xi_{01}$. Then it is easy to see that $d\omega = 0$ and $[(\omega,\omega,0)] = [\xi]$. The injectivity of α is not difficult to show.

We define the "cup product"

$$A^p(\mathcal{U}) \times A^q(\mathcal{U}) \longrightarrow A^{p+q}(\mathcal{U})$$

by assigning to ξ in $A^p(\mathcal{U})$ and η in $A^q(\mathcal{U})$ the element $\xi \smile \eta$ in $A^{p+q}(\mathcal{U})$ given by

$$(\xi \smile \eta)_i = \xi_i \wedge \eta_i,\ i = 0,\ 1, \quad (\xi \smile \eta)_{01} = (-1)^p \xi_0 \wedge \eta_{01} + \xi_{01} \wedge \eta_1. \quad (1.5.2)$$

Then we have $D(\xi \smile \eta) = D\xi \smile \eta + (-1)^p \xi \smile D\eta$. Thus it induces the cup product

$$H^p_D(\mathcal{U}) \times H^q_D(\mathcal{U}) \longrightarrow H^{p+q}_D(\mathcal{U})$$

compatible, via the isomorphism of 1.5.1, with the cup product in the de Rham cohomology.

1.5.1 Integration on the Čech-de Rham Cohomology

Now we recall the integration on the Čech-de Rham cohomology (cf. [109]). Suppose that the m'-dimensional manifold M is oriented and compact and let $\mathcal{U} = \{U_0, U_1\}$ be a covering of M. Let $R_0, R_1 \subset M$ be two compact manifolds of dimension m' with C^∞ boundary with the following properties:
(1) $R_j \subset U_j$ for $j = 0, 1$,
(2) $\mathrm{Int}R_0 \cap \mathrm{Int}R_1 = \emptyset$ and
(3) $R_0 \cup R_1 = M$.

Let $R_{01} = R_0 \cap R_1$ and give R_{01} the orientation as the boundary of R_0; $R_{01} = \partial R_0$, equivalently give R_{01} the orientation opposite to that of the boundary of R_1; $R_{01} = -\partial R_1$. We define the integration

$$\int_M : A^{m'}(\mathcal{U}) \longrightarrow \mathbb{C} \quad \text{by} \quad \int_M \xi = \int_{R_0} \xi_0 + \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}.$$

Then by the Stokes theorem, we see that if $D\xi = 0$ then $\int_M \xi$ is independent of $\{R_0, R_1\}$ and that if $\xi = D\eta$ for some $\eta \in A^{p-1}(\mathcal{U})$ then $\int_M \xi = 0$. Thus we may define the integration

$$\int_M : H^{m'}_D(\mathcal{U}) \longrightarrow \mathbb{C},$$

which is compatible with the integration on the de Rham cohomology via the isomorphism of 1.5.1.

1.5.2 Relative Čech-de Rham Cohomology – Alexander Duality

Next we define the relative Čech-de Rham cohomology and describe the Alexander duality. Let M be an m'-dimensional oriented manifold (not necessarily compact) and S a compact subset of M. Let $U_0 = M \setminus S$ and let U_1 be an open neighborhood of S. We consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We set

$$A^p(\mathcal{U}, U_0) = \{\, \xi = (\xi_0, \xi_1, \xi_{01}) \in A^p(\mathcal{U}) \mid \xi_0 = 0 \,\}.$$

Then we see that if ξ is in $A^p(\mathcal{U}, U_0)$, $D\xi$ is in $A^{p+1}(\mathcal{U}, U_0)$. This gives rise to another complex, called the relative Čech-de Rham complex, and we may define the p-th relative Čech-de Rham cohomology of the pair $(\mathcal{U}, U_0)$ as

$$H^p_D(\mathcal{U}, U_0) = \mathrm{Ker} D^p / \mathrm{Im} D^{p-1}.$$

By the five lemma, we see that there is a natural isomorphism

$$H^p_D(\mathcal{U}, U_0) \simeq H^p(M, M \setminus S; \mathbb{C}).$$

Let R_1 be a compact manifold of dimension m' with C^∞ boundary such that $S \subset \mathrm{Int} R_1 \subset R_1 \subset U_1$. Let $R_0 = M \setminus \mathrm{Int} R_1$. Note that $R_0 \subset U_0$. The integral operator $\int_M$ (which is not defined in general for $A^{m'}(\mathcal{U})$ unless M is compact) is well defined on $A^{m'}(\mathcal{U}, U_0)$:

$$\int_M : A^{m'}(\mathcal{U}, U_0) \longrightarrow \mathbb{C}, \qquad \int_M \xi = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01},$$

and induces an operator $\int_M : H^{m'}_D(\mathcal{U}, U_0) \to \mathbb{C}$.

In the cup product $A^p(\mathcal{U}) \times A^{m'-p}(\mathcal{U}) \to A^{m'}(\mathcal{U})$ given as (1.5.2), we see that if $\xi_0 = 0$, the right hand side depends only on ξ_1, ξ_{01}, and η_1. Thus we have a pairing $A^p(\mathcal{U}, U_0) \times A^{m'-p}(U_1) \to A^{m'}(\mathcal{U}, U_0)$, which, followed by the integration, gives a bilinear pairing

$$A^p(\mathcal{U}, U_0) \times A^{m'-p}(U_1) \longrightarrow \mathbb{C}.$$

If we further assume that U_1 is a regular neighborhood of S, this induces the Alexander duality (cf 1.3.2 and [25])

$$A : H^p(M, M \setminus S; \mathbb{C}) \simeq H^p_D(\mathcal{U}, U_0) \xrightarrow{\sim} H^{m'-p}(U_1, \mathbb{C})^* \simeq H_{m'-p}(S, \mathbb{C}). \tag{1.5.3}$$

Proposition 1.5.1. *[25] If M is compact, we have the commutative diagram*

$$\begin{array}{ccc} H^p(M, M \setminus S; \mathbb{C}) & \xrightarrow{j^*} & H^p(M, \mathbb{C}) \\ \wr\big\downarrow A & & \wr\big\downarrow P \\ H_{m'-p}(S, \mathbb{C}) & \xrightarrow{i_*} & H_{m'-p}(M, \mathbb{C}), \end{array}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow M$ and $(M, \emptyset) \hookrightarrow (M, M \setminus S)$.

We finish this section by giving a fundamental example of computation of relative Čech-de Rham cohomology.

Example 1.5.1. Let $M = \mathbb{R}^{m'}$ and $S = \{0\}$ with $m' \geq 2$. In this case, $U_0 = \mathbb{R}^{m'} \setminus \{0\}$, which retracts to $\mathbb{S}^{m'-1}$. Let $U_1 = \mathbb{R}^{m'}$. In this situation, we compute $H^p_D(\mathcal{U}, U_0)$. For $p = 0$, each element ξ in $A^0(\mathcal{U}, U_0)$ can be written as $\xi = (0, f, 0)$ for some C^∞ function f on U_1. If $D\xi = 0$, we have $f \equiv 0$ and therefore $H^0_D(\mathcal{U}, U_0) = \{0\}$. Next, an element ξ in $A^1(\mathcal{U}, U_0)$ can be written as $\xi = (0, \xi_1, f)$ with ξ_1 a 1-form on U_1 and f a C^∞ function on $U_0 \cap U_1$. If ξ is a cocycle then $d\xi_1 = 0$ on U_1 and $df = \xi_1$ on $U_0 \cap U_1$. By the Poincaré lemma the first condition implies that $\xi_1 = dg$ for some C^∞ function g on U_1 and the second condition implies that $f \equiv g + c$ for some $c \in \mathbb{C}$. Therefore f has a C^∞ extension, still denoted by f, over $\{0\}$ and $\xi = (0, df, f) = D(0, f, 0)$. Hence $H^1_D(\mathcal{U}, U_0) = \{0\}$. For $p \geq 2$ the map

$$H^{p-1}_{dR}(U_0) \longrightarrow H^p_D(\mathcal{U}, U_0) \qquad \text{given by} \quad [\omega] \mapsto [(0, 0, -\omega)]$$

can be shown to be an isomorphism (we leave the details to the reader) and we have

$$H^p_D(\mathcal{U}, U_0) \simeq H^{p-1}_{dR}(U_0) \simeq H^{p-1}(\mathbb{S}^{m'-1}) = \begin{cases} \mathbb{C}, & \text{for} \quad p = m', \\ 0, & \text{for} \quad p = 2, \ldots, m'-1. \end{cases}$$

An explicit generator of $H^{m'-1}(\mathbb{S}^{m'-1})$ is given as follows ([75, p. 370]). For $x = (x_1, \ldots, x_{m'})$ in $\mathbb{R}^{m'}$, we set $\Phi(x) = dx_1 \wedge \cdots \wedge dx_{m'}$ and

$$\Phi_i(x) = (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{m'}.$$

Also, let $C_{m'}$ be the constant given by

$$C_{m'} = \begin{cases} \frac{(\ell-1)!}{2\pi^\ell}, & \text{for } m' = 2\ell \\ \frac{(2\ell)!}{2^{2\ell+1}\pi^\ell \ell!}, & \text{for } m' = 2\ell + 1. \end{cases}$$

Then the form

$$\psi_{m'} = C_{m'} \frac{\sum_{i=1}^{m'} \Phi_i(x)}{\|x\|^{m'}}$$

is a closed $(m'-1)$-form on $\mathbb{R}^{m'} \setminus 0$ whose integral on the unit sphere $\mathbb{S}^{m'-1}$ (in fact a sphere of arbitrary radius) is 1. Now we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$, then $\psi_{2m} = (\beta_m + \overline{\beta_m})/2$, where

$$\beta_m = C'_m \frac{\sum_{i=1}^{m} \overline{\Phi_i(z)} \wedge \Phi(z)}{\|z\|^{2m}}, \qquad C'_m = (-1)^{\frac{m(m-1)}{2}} \frac{(m-1)!}{(2\pi\sqrt{-1})^m}. \quad (1.5.4)$$

Then β_m is a closed $(m, m-1)$-form on $\mathbb{C}^m \setminus 0$, real on $\mathbb{S}^{2m-1}$ and $\int_{\mathbb{S}^{2m-1}} \beta_m = 1$. We call β_m the Bochner–Martinelli kernel on $\mathbb{C}^m$. Note that

$$\beta_1 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z},$$

is the Cauchy kernel on $\mathbb{C}$.

1.6 Localization of Chern Classes

In a previous section we described the topological viewpoint for localizing Chern classes on a given compact subset S of a manifold M, taking an appropriate frame in the appropriate skeleton of a neighborhood of S. This gives an explicit representative of the Chern class which represents it as a relative cohomology class, with a specific contribution localized at S. We also know (see for instance [14,19,123]) that Chern classes of manifolds and vector bundles in general can be defined via Chern–Weil theory, using the curvature tensor of a connection. To describe the localization of Chern classes, we modify the Chern–Weil theory so that it is adapted to the Čech-de Rham cohomology.

1.6.1 Characteristic Classes in the Čech-de Rham Cohomology

Let M be a C^∞ manifold and $\mathcal{U} = \{U_0, U_1\}$ an open covering of M. For a vector bundle E over M, we take a connection ∇_j on U_j, $j = 0, 1$, and let $c^i(\nabla_*)$ be the element of $A^{2i}(\mathcal{U})$ given by

$$c^i(\nabla_*) = (c^i(\nabla_0), c^i(\nabla_1), c^i(\nabla_0, \nabla_1)). \tag{1.6.1}$$

Then we see that $Dc^i(\nabla_*) = 0$ and this defines a class $[c^i(\nabla_*)]$ in $H_D^{2i}(\mathcal{U})$. It is not difficult to show the following

Theorem 1.6.2. *The class $[c^i(\nabla_*)] \in H_D^{2i}(\mathcal{U})$ corresponds to the Chern class $c^i(E) \in H_{dR}^{2i}(M)$ under the isomorphism of Theorem 1.5.1.*

By a similar construction, we may define the characteristic class $\varphi(E)$ for a polynomial φ in the Chern polynomials in the Čech-de Rham cohomology. It can be done also for virtual bundles (see Chap. 5).

Using Bott difference forms, we may define characteristic classes in the Čech-de Rham cohomology for an arbitrary open covering of M.

This way of representing characteristic classes is particularly useful in dealing with the "localization problem," which we explain in the next subsection. This theory involves vanishing theorems, one of which is given as follows.

Let E be a complex vector bundle of rank k on a C^∞ manifold M. Let $\mathbf{s} = (s_1, \ldots, s_r)$ be an r-frame of E on an open set U. Recall that (Definition 1.4.2) a connection ∇ for E on U is $\mathbf{s}$-trivial, if $\nabla(s_i) = 0$ for $i = 1, \ldots, r$.

Proposition 1.6.1. *If ∇ is $\mathbf{s}$-trivial, then*

$$c^j(\nabla) \equiv 0 \qquad \textit{for} \quad j \geq k - r + 1.$$

Proof. For simplicity, we prove the proposition when $r = 1$. Let $U \subset M$ be an open set such that $E|_U \simeq U \times \mathbb{C}^k$. Since $s_1 \neq 0$ everywhere on M, we may take a frame $\mathbf{e} = (e_1, \ldots, e_k)$ on U so that $e_1 = s_1$. Then all the entries of the first row of the curvature matrix κ of ∇ with respect to $\mathbf{e}$ are zero. Since $c^k(\nabla) = \det \kappa$, up to a constant, we have $c^k(\nabla) = 0$.

1.6.2 Localization of Characteristic Classes of Complex Vector Bundles

In this subsection, we explain how we obtain indices and residues in the subsequent sections.

Let M be an oriented C^∞ manifold of dimension m' and E a C^∞ complex vector bundle of rank k over M. Also, let S be a closed set in M and U_1 a neighborhood of S in M. Setting $U_0 = M \setminus S$, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. For a homogeneous symmetric polynomial φ of degree d, the characteristic class $\varphi(E)$ is represented by the cocycle $\varphi(\nabla_*)$ in $A^{2d}(\mathcal{U})$ given by

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1 respectively. Sometimes, it happens that we have a "vanishing theorem" on U_0 for some polynomials φ. Namely, there is some "geometric object" γ on U_0, to which is associated a class $\mathcal{C}$ of connections for E on U_0 such that, for a connection ∇_0 belonging to $\mathcal{C}$ and for a certain polynomial φ, we have

$$\varphi(\nabla_0) \equiv 0.$$

We call a connection belonging to $\mathcal{C}$ *special* and a polynomial φ as above *adapted* to γ. As we see below, this kind of vanishing defines a localization of the relevant characteristic class. Moreover, if we have also the vanishing of the Bott difference forms for families of special connections, we may show that the localization does not depend on the connections involved. This is the case in all the cases we consider below and we assume this hereafter.

Thus, if ∇_0 is special and if φ is adapted to γ, then the above cocycle $\varphi(\nabla_*)$ is in $A^{2d}(\mathcal{U}, U_0)$ and it defines a class in $H^{2d}(M, M \setminus S; \mathbb{C})$, which is denoted by $\varphi_S(E, \gamma)$. It is sent to the class $\varphi(E)$ by the canonical homomorphism $j^* : H^{2d}(M, M \setminus S; \mathbb{C}) \to H^{2d}(M, \mathbb{C})$. It is not difficult to see that the class $\varphi_S(E, \gamma)$ does not depend on the choice of the special connection ∇_0 or the connection ∇_1.

We call $\varphi_S(E, \gamma)$ the *localization of* $\varphi(E)$ *at* S *by* γ. Suppose S is a compact set admitting a regular neighborhood. Then we have the Alexander duality (1.5.3)

$$A : H^{2d}(M, M \setminus S; \mathbb{C}) \xrightarrow{\sim} H_{m'-2d}(S, \mathbb{C}).$$

Thus the class $\varphi_S(E, \gamma)$ defines a class in $H_{m'-2d}(S, \mathbb{C})$, which we call the residue of γ for the class $\varphi(E)$ at S and denote by $\mathrm{Res}_\varphi(\gamma, E; S)$.

We suppose that U_1 is a regular neighborhood and let R_1 be an m'-dimensional manifold with C^∞ boundary in U_1 containing S in its interior and we set $R_{01} = -\partial R_1$. Then the residue $\mathrm{Res}_\varphi(\gamma, E; S)$ is represented by an $(m' - 2d)$-cycle C in S such that

$$\int_C \eta = \int_{R_1} \varphi(\nabla_1) \wedge \eta + \int_{R_{01}} \varphi(\nabla_0, \nabla_1) \wedge \eta \qquad (1.6.3)$$

for every closed $(m' - 2d)$-form η on U_1. In particular, if $2d = m'$, the residue is a complex number given by

$$\mathrm{Res}_\varphi(\gamma, E; S) = \int_{R_1} \varphi(\nabla_1) + \int_{R_{01}} \varphi(\nabla_0, \nabla_1). \qquad (1.6.4)$$

Suppose moreover that S has a finite number of connected components $(S_\lambda)_\lambda$. Then we have a decomposition

$$H_{m'-2d}(S, \mathbb{C}) = \bigoplus_\lambda H_{m'-2d}(S_\lambda, \mathbb{C})$$

and accordingly, we have the residue $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ in $H_{m'-2d}(S_\lambda, \mathbb{C})$ for each λ. Replacing U_1 by a regular neighborhood U_λ of S_λ, disjoint from the other components, and R_1 by an m'-dimensional manifold R_λ with boundary in U_λ containing S_λ in its interior, we have an expression $(1.6.3)_\lambda$ or $(1.6.4)_\lambda$ for the residue $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ similar to (1.6.3) or (1.6.4).

From the above considerations and Proposition 1.5.1, we have the following "residue theorem."

Theorem 1.6.5. *In the above situation,*

(1) *For each connected component S_λ of S, we have the residue* $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ *in the homology $H_{m'-2d}(S_\lambda, \mathbb{C})$, which is determined by the local behavior of γ near S_λ and is expressed as $(1.6.3)_\lambda$ or $(1.6.4)_\lambda$.*

(2) *If M is compact,*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_\varphi(\gamma, E; S_\lambda) = \varphi(E) \frown [M] \qquad \text{in} \ \ H_{m'-2d}(M, \mathbb{C}),$$

where $i_\lambda : S_\lambda \hookrightarrow M$ denotes the inclusion.

Remark 1.6.1. If $2d = m'$, we do not have to assume that S admits a regular neighborhood. Simply take an arbitrary open neighborhood as U_1 and define $\mathrm{Res}_\varphi(\gamma, E; S)$ by (1.6.4) with R_1 as above, then Theorem 1.6.5 is still valid.

1.6.3 Localization of the Top Chern Class

Let E be a C^∞ complex vector bundle of rank k over an oriented C^∞ manifold M of dimension m'. Let s be a nonvanishing section of E on some open set U. Recall that a connection ∇ for E on U is s-trivial, if $\nabla(s) = 0$. If ∇ is an s-trivial connection, we have the vanishing (Proposition 1.6.1)

$$c^k(\nabla) = 0. \tag{1.6.6}$$

Let S be a closed set in M and suppose we have a C^∞ nonvanishing section s of E on $M \setminus S$. Then, from the above fact, applying the arguments in Sect. 1.6.2 taking c^k as φ and s-trivial connections as special connections, we see that there is a natural lifting $c^k(E, s)$ in $H^{2k}(M, M \setminus S; \mathbb{C})$ of the top Chern class $c^k(E)$ in $H^{2k}(M, \mathbb{C})$. We call $c^k(E, s)$ the localization of $c^k(E)$ with respect to the section s at S.

Also, if S is a compact set admitting a regular neighborhood, the class $c^k(E, s)$ defines a class in $H_{m'-2k}(S, \mathbb{C})$, which we call the residue of s for E at S with respect to c^k and denote by $\mathrm{Res}_{c^k}(s, E; S)$. This residue corresponds to what is called the "localized top Chern class" of E with respect to s in [59, Sect. 14.1].

The residue $\mathrm{Res}_{c^k}(s, E; S)$ is represented by an $(m' - 2k)$-cycle C in S_λ satisfying (1.6.3). In particular, if $2k = m'$, the residue is a complex number given by (1.6.4) with $\varphi = c^k$. If S has a finite number of connected components $(S_\lambda)_\lambda$, we have the residue $\mathrm{Res}_{c^k}(s, E; S_\lambda)$ in $H_{m'-2k}(S_\lambda, \mathbb{C})$ for each λ. Moreover, Theorem 1.6.5 becomes

Theorem 1.6.7. *In the above situation,*

(1) *For each connected component S_λ of S, we have the residue* $\mathrm{Res}_{c^k}(s, E; S_\lambda)$ *in the homology $H_{m'-2k}(S_\lambda, \mathbb{C})$.*

(2) *If M is compact,*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_{c^k}(s, E; S_\lambda) = c^k(E) \frown [M] \qquad \text{in} \;\; H_{m'-2k}(M, \mathbb{C}).$$

Remark 1.6.2. 1. In fact it can be shown that the above residues are in the integral homology and the equality in Theorem 1.6.7 holds in the integral homology (see [161]).
2. A localization theory of Chern classes, other than the top one, by a finite number of sections can be developed similarly (see [159–161]).

1.6.4 Hyperplane Bundle

As a basic example of the theory developed in the previous subsections, we prove that the Poincaré dual of the first Chern of the hyperplane bundle L_H on a projective space is (the homology class of) the hyperplane H. In fact, we prove a more precise statement that the Alexander dual of the localization of the first Chern of L_H by the canonical section is the fundamental class of H in the homology of H. Note that the essential point in the proof is the Cauchy integral formula; $\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{dz}{z} = 1$.

Let $\mathbb{CP}^m$ be the m-dimensional complex projective space with homogeneous coordinates $[\zeta_0, \dots, \zeta_m]$. We denote by W_i the open set in $\mathbb{CP}^m$ defined by $\zeta_i \neq 0$, $i = 0, \dots, m$. Let H denote the hyperplane defined by $\zeta_0 = 0$ and L_H the line bundle determined by H. Recall that L_H is defined by the system of transition functions h_{ij}, $h_{ij} = \zeta_j/\zeta_i$. The canonical section s is represented by the collection (s_i), where s_i is a holomorphic function on W_i given by $s_i = \zeta_0/\zeta_i$. Since the zero set of s is H, we have the localization $c^1(L_H, s)$ of $c^1(L_H)$ in $H^2(\mathbb{CP}^m, \mathbb{CP}^m \setminus H)$.

Theorem 1.6.8. *The image of $c^1(L_H, s)$ by the Alexander isomorphism*

$$H^2(\mathbb{CP}^m, \mathbb{CP}^m \setminus H) \xrightarrow{\sim} H_{2m-2}(H)$$

is the fundamental class $[H]$, i.e., $\mathrm{Res}_{c^1}(s, L_H; H) = [H]$.

Proof. Let $\mathcal{U} = \{U_0, U_1\}$ be the covering of $\mathbb{CP}^m$ consisting of $U_0 = \mathbb{CP}^m \setminus H$ and a tubular neighborhood U_1 of H with a C^∞ retraction $\rho : U_1 \to H$. Let ∇_0 be an s-trivial connection for L_H on U_0 so that $c^1(\nabla_0) = 0$ and ∇_1 an arbitrary connection for L_H on U_1. Then the class $c^1(L_H, s)$ is represented by the cocycle $(0, c^1(\nabla_1), c^1(\nabla_0, \nabla_1))$ in $A^2(\mathcal{U}, U_0)$. Let R_1 be a closed tubular neighborhood of H in U_1 and $R_{01} = -\partial R_1$. Our aim is to show that (cf. (1.6.3))

$$\int_H \eta = \int_{R_1} c^1(\nabla_1) \wedge \eta + \int_{R_{01}} c^1(\nabla_0, \nabla_1) \wedge \eta \qquad (1.6.9)$$

for every closed $(2m-2)$-form η on U_1.

Since the retraction map ρ induces an isomorphism $\rho^* : H^{2m-2}_{dR}(H) \xrightarrow{\sim} H^{2m-2}_{dR}(U_1)$, we see that there exist a closed $(2m-2)$-form θ on H and a $(2m-3)$-form τ on U_1 with $\eta = \rho^*\theta + d\tau$. By the Stokes theorem and the property of the difference form $c^1(\nabla_0, \nabla_1)$, we see that it suffices to prove (1.6.9) for $\eta = \rho^*\theta$. For the left hand side, we have $\int_H \rho^*\theta = \int_H \theta$. To compute the right hand side, we note that $L_H|_{U_1} \simeq \rho^*(L_H|_H)$. Let ∇ be a connection for $L_H|_H$ and take as ∇_1 the connection corresponding to $\rho^*\nabla$. Then we have $c^1(\nabla_1) \wedge \rho^*\theta = \rho^*(c^1(\nabla) \wedge \theta) = 0$, since $c^1(\nabla) \wedge \theta$ is a $2m$-form on H. In the second term of the right hand side, R_{01} is an S^1 bundle over H with the orientation opposite to the natural one. Let ρ_{01} denote the restriction of ρ to R_{01}. Then by the projection formula, we have

$$\int_{R_{01}} c^1(\nabla_0, \nabla_1) \wedge \rho^*\theta = -\int_H (\rho_{01})_* c^1(\nabla_0, \nabla_1) \cdot \theta,$$

where $(\rho_{01})_*$ denotes the integration along the fiber of ρ_{01} so that the form $(\rho_{01})_* c^1(\nabla_0, \nabla_1)$ is in fact a function on H. It suffices to prove that this function is identically equal to -1. Let p be an arbitrary point in H and suppose it is in W_i, $i \neq 0$. In the sequel, we identify $L_H|_{W_i}$ with $W_i \times \mathbb{C}$. On W_i, the section s is represented by the function $s_i = \zeta_0/\zeta_i$, which can also be thought of as a fiber coordinate of the retraction ρ. Let ∇' denote the connection for $L_H|_H$ on $W_i \cap H$ trivial with respect to the frame ℓ given by $\ell(q) = (q, 1)$ for q in $W_i \cap H$. We may modify ∇' away from a neighborhood of p to obtain a connection ∇ for $L_H|_H$ on H. The pullback $\nabla_1 = \rho^*\nabla$ is a connection for L_H which is trivial with respect to the frame ℓ_1 given by $\ell_1(q) = (q, 1)$ for q in a neighborhood W of p in W_i. Now we try to find $c^1(\nabla_0, \nabla_1)$ on $W \cap U_{01} = W \setminus H$ (cf. (1.4.3)). For this, let $\tilde{\nabla}$ be the connection for $L_H \times \mathbb{R}$ given by $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$. Let θ_i be the connection form of ∇_i with respect to the frame ℓ_1, $i = 0, 1$. Then $\theta_1 = 0$ and, since θ_0 is s-trivial and $\ell_1 = \frac{1}{z}s$, $z = \zeta_0/\zeta_i$, by (1.4.1), we have $\theta_0 = d\left(\frac{1}{z}\right)/\frac{1}{z} = -\frac{dz}{z}$. Thus the connection form $\tilde{\theta}$ of $\tilde{\nabla}$ is given by

$$\tilde{\theta} = -(1-t)\frac{dz}{z}.$$

Hence the curvature form $\tilde{\kappa}$ of $\tilde{\nabla}$ is given by $\tilde{\kappa} = d\tilde{\theta} = dt \wedge \frac{dz}{z}$ and we get

$$c^1(\nabla_0, \nabla_1) = \frac{\sqrt{-1}}{2\pi}\pi_*\tilde{\kappa} = -\frac{1}{2\pi\sqrt{-1}}\frac{dz}{z},$$

where π_* denotes the integration along the fiber of the projection map $\pi : W \setminus H \times [0,1] \to W \setminus H$. Therefore, by the Cauchy integral formula, we have

$$(\rho_{01})_* c^1(\nabla_0, \nabla_1) = -1$$

in a neighborhood of p.

See [157] and [161] for more general results and thorough discussions in this direction.

1.6.5 Grothendieck Residues

As we have seen in the previous subsection and will see also in the sequel, the residues of characteristic classes are deeply related to Grothendieck residues. In this subsection, we briefly review this subject. For details, we refer to, e.g., [75].

Let U be a neighborhood of the origin 0 in $\mathbb{C}^m$ and $f_1, \dots, f_m$ holomorphic functions on U such that their common set of zeros consists only of 0. For a holomorphic m-form ω on U, we set

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_m \end{bmatrix} = \frac{1}{(2\pi\sqrt{-1})^m} \int_\Gamma \frac{\omega}{f_1 \cdots f_m}, \qquad (1.6.10)$$

where Γ is an m-cycle in U defined by

$$\Gamma = \{\, z \in U \mid |f_1(z)| = \cdots = |f_m(z)| = \varepsilon \,\}$$

for a small positive number ε. We orient Γ so that the form $d\theta_1 \wedge \cdots \wedge d\theta_m$ is positive, $\theta_i = \arg f_i$.

Example 1.6.1. If $m = 1$, the above residue 1.6.10 is the usual Cauchy residue at 0 of the meromorphic 1-form ω/f_1.

Example 1.6.2. In the next subsection, we give various expressions for the residue of the top Chern class at an isolated singularity of a section s. If $(f_1, \dots, f_m)$ denote local components of s around the singularity, the Grothendieck residue with $\omega = df_1 \wedge \cdots \wedge df_m$ appears as an "analytic expression" of the residue. Thus we have

$$\operatorname{Res}_0 \begin{bmatrix} df_1 \wedge \cdots \wedge df_m \\ f_1, \ldots, f_m \end{bmatrix} = \dim_{\mathbb{C}} \mathcal{O}_m/(f_1, \ldots, f_m) = \operatorname{Ind}_{\mathrm{PH}}(v, 0), \qquad (1.6.11)$$

where v denotes the holomorphic vector field $\sum_{i=1}^m f_i \cdot \partial/\partial z_i$. This positive integer is also interpreted as the intersection number $(D_1 \cdots D_m)_0$ at 0 of the divisors D_i defined by f_i (cf. [75, Ch.5, 2], [157]).

Example 1.6.3. In particular, if $f_i = \partial f/\partial z_i$ for some f in $\mathcal{O}_m$, then the residue is the Milnor number $\mu(V, 0)$ of the hypersurface V defined by f at 0;

$$\operatorname{Res}_0 \begin{bmatrix} d\left(\frac{\partial f}{\partial z_1}\right) \wedge \cdots \wedge d\left(\frac{\partial f}{\partial z_m}\right) \\ \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_m} \end{bmatrix} = \mu(V, 0).$$

We also call this number the multiplicity of f at 0 and denote it by $m(f, 0)$ (cf. Sect. 1.6.7-b below).

1.6.6 Residues at an Isolated Zero

Let E be a holomorphic vector bundle of rank m over a complex manifold M of dimension m. Suppose we have a section s with an isolated zero at p in M. In this situation, we have $\operatorname{Res}_{c^m}(s, E; p)$ in $H_0(\{p\}, \mathbb{C}) = \mathbb{C}$. In the following, we give explicit expressions of this residue.

Let U be an open neighborhood of p where the bundle E is trivial with holomorphic frame $(e_1, \ldots, e_m)$. We write $s = \sum_{i=1}^m f_i\, e_i$ with f_i holomorphic functions on U.

(I) Analytic expression

Theorem 1.6.12. *In the above situation, we have*

$$\operatorname{Res}_{c^m}(s, E; p) = \operatorname{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_m \\ f_1, \ldots, f_m \end{bmatrix}.$$

Proof. We indicate the proof for the case $m = 1$ (for $m > 1$, we use the Čech-de Rham cohomology theory for m open sets, see [157], [160]). Thus $s = f e_1$ for some holomorphic function f on U. Let R be a closed disk about p in U. In the expression (1.6.4) of the residue, we may take as ∇_1 an e_1-trivial connection on U, thus $c^1(\nabla_1) \equiv 0$ and

$$\operatorname{Res}_p(s, E; p) = -\int_{\partial R} c^1(\nabla_0, \nabla_1)$$

with ∇_0 an s-trivial connection on $U' = U \setminus \{p\}$. The Bott difference form $c^1(\nabla_0, \nabla_1)$ can be computed as in the proof of Theorem 1.6.8. If we let θ_i be the connection matrix of ∇_i, $i = 0, 1$, with respect to the frame e_1, we have $\theta_1 = 0$ and $\theta_0 = -\frac{df}{f}$. Thus this time we have

$$c^1(\nabla_0, \nabla_1) = -\frac{1}{2\pi\sqrt{-1}} \frac{df}{f},$$

which proves the theorem (for the case $m = 1$).

Remark 1.6.3. For general m, if we take suitable connections we see that the difference form is given by

$$c^m(\nabla_0, \nabla_1) = -f^* \beta_m,$$

where $f = (f_1, \ldots, f_m)$ and β_m denotes the Bochner–Martinelli kernel on $\mathbb{C}^m$ (cf. (1.5.4)). This gives a direct proof of Theorem 1.6.14 below. Thus we reprove the fact that the Grothendieck residue in the above theorem is equal to the mapping degree of f (cf. [75, Ch.5, 1. Lemma]).

(II) Algebraic expression

Theorem 1.6.13. *In the above situation, we have*

$$\mathrm{Res}_{c^m}(s, E; p) = \dim \mathcal{O}_m/(f_1, \ldots, f_m).$$

This can be proved, for example, by perturbing the sections and using the theory of Cohen–Macaulay rings (e.g., [160]).

(III) Topological Expression

Let $\mathbb{S}^{2m-1}_\varepsilon$ denote a small $2m-1$ sphere in U with center p. Then we have the mapping

$$\varphi = \frac{f}{\|f\|} : \mathbb{S}^{2m-1}_\varepsilon \longrightarrow \mathbb{S}^{2m-1},$$

where $\mathbb{S}^{2m-1}$ denotes the unit sphere in $\mathbb{C}^m$.

Theorem 1.6.14. *In the above situation, we have*

$$\mathrm{Res}_{c^m}(s, E; p) = \deg \varphi.$$

This can also be proved by perturbing the sections, see [75], [160].

Remark 1.6.4. There are similar expressions as above for the residues of vector bundles on singular varieties with respect to an appropriate number of sections (see [160]).

1.6.7 *Examples*

(a) Poincaré–Hopf Index Theorem

Let M be a complex manifold of dimension m. We take as E the holomorphic tangent bundle TM. Then a section of TM is a (complex) vector field v. One can check (see, e.g., [161]) that the Poincaré–Hopf index $\mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda)$ of v at a connected component S_λ of its zero set S, that we defined in 1.1.3 can be expressed as

$$\mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda) = \mathrm{Res}_{c^m}(v, TM; S_\lambda).$$

Then, if M is compact, by Theorem 1.6.7, we have

$$\sum_\lambda \mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda) = \int_M c^m(M),$$

where $c^m(M) = c^m(TM)$ and it is known that the right hand side coincides with the Euler–Poincaré characteristic $\chi(M)$ of M ("Gauss–Bonnet formula"). Thus, by Theorem 1.6.7, we recover the Poincaré–Hopf theorem in case v is holomorphic and the zeros are isolated.

(b) Multiplicity Formula

Let M be a complex manifold of dimension m. We take as E the holomorphic cotangent bundle T^*M. For a holomorphic function f on M, its differential df is a section of T^*M. The zero set S of df coincides with the critical set $C(f)$ of f. We define the multiplicity $m(f, S_\lambda)$ of f at a connected component S_λ of $C(f)$ by

$$m(f, S_\lambda) = \mathrm{Res}_{c^m}(df, T^*M; S_\lambda).$$

Note that, if S_λ consists of a point p, it coincides with the multiplicity $m(f, p)$ of f at p described in Example 1.6.3.

Now we consider the global situation. Let $f : M \to C$ be a holomorphic map of M onto a complex curve (Riemann surface) C. The differential

$$df : TM \longrightarrow f^*TC$$

of f determines a section of $T^*M \otimes f^*TC$, which is also denoted by df. The set of zeros of df is the critical set $C(f)$ of f. Suppose $C(f)$ is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then we have the residue $\mathrm{Res}_{c^m}(df, T^*M \otimes f^*TC; S_\lambda)$ for each λ. If M is compact, by Theorem 1.6.7, we have

$$\sum_\lambda \mathrm{Res}_{c^m}(df, T^*M \otimes f^*TC; S_\lambda) = \int_M c^m(T^*M \otimes f^*TC).$$

We look at the both sides of the above more closely. In the sequel, we set $D(f) = f(C(f))$, the set of critical values. Then, if M is compact, f defines a C^∞ fiber bundle structure on $M \setminus C(f) \to C \setminus D(f)$.

We refer to [87] for a precise proof of the following

Lemma 1.6.1. *If M is compact, and if $D(f)$ consists of isolated points,*

$$\int_M c^m(T^*M \otimes f^*TC) = (-1)^m(\chi(M) - \chi(\mathbf{F})\,\chi(C)),$$

where $\mathbf{F}$ denotes a general fiber of f.

Suppose that $f(S_\lambda)$ is a point. Taking a coordinate on C around $f(S_\lambda)$, we think of f as a holomorphic function near S_λ. Then we may write

$$\mathrm{Res}_{c^m}(df, T^*M \otimes f^*TC; S_\lambda) = \mathrm{Res}_{c^m}(df, T^*M; S_\lambda) = m(f, S_\lambda),$$

the multiplicity of f at S_λ. Thus we have

Theorem 1.6.15. *Let $f : M \to C$ be a holomorphic map of a compact complex manifold M of dimension m onto a complex curve C. If the critical values $D(f)$ of f consists of only isolated points, then*

$$\sum_\lambda m(f, S_\lambda) = (-1)^m(\chi(M) - \chi(\mathbf{F})\,\chi(C)),$$

where the sum is taken over the connected components S_λ of $C(f)$.

In particular, we have ([86], see also [59, Example 14.1.5]):

Corollary 1.6.1. *In the above situation, if the critical set $C(f)$ of f consists of only isolated points,*

$$\sum_{p \in C(f)} m(f, p) = (-1)^m(\chi(M) - \chi(\mathbf{F})\,\chi(C)).$$

See [87] for the definition of multiplicities of functions on possibly singular varieties and formula similar to the above for these multiplicities.

Chapter 2
The Schwartz Index

Abstract The index of a tangent vector field in a singular point is well-defined on manifolds, as described in the previous chapter. When working with singular analytic varieties, it is necessary to give a sense to the notion of "tangent" vector field and, once this is done, it is natural to ask what should be the notion of "the index" at a singularity of the suitable vector field. Indices of vector fields on singular varieties were first considered by M.-H. Schwartz in [139, 141] (see also [33, 142]) in her study of the Poincaré–Hopf Theorem and Chern classes for singular varieties. For her purpose there was no point in considering vector fields in general, but only a special class of vector fields that she called "radial," which are obtained by the important process of *radial extension*. In this chapter we explain the definition of the corresponding index as it was defined by M.-H. Schwartz for vector fields constructed by radial extension. Complete description and constructions will be found in [28].

We define a natural extension of this index for arbitrary (stratified) vector fields on singular varieties. This index is sometimes called "radial index" in the literature, but we prefer to call it here *the Schwartz index*. The Schwartz index for arbitrary stratified vector fields was first defined by H. King and D. Trotman in [96], and later independently in [6, 49, 149]. In [30, 31] this index was interpreted in differential-geometric terms and this was used to study its relations with various characteristic classes for singular varieties. This is discussed in [28] and in Chap. 10 below.

2.1 Isolated Singularity Case

Consider first the case where the space is a complex analytic variety $V \subset \mathbb{C}^m$ of dimension $n > 1$ with an isolated singularity at 0. Let U be an open ball around $0 \in \mathbb{C}^m$, small enough so that every sphere in U centered at 0 meets V transversally (see [120]). For simplicity we restrict the discussion to U. Let v_{rad} be a continuous vector field on $V \setminus \{0\}$ which is transverse (outwards-pointing) to all spheres $\mathbb{S}_\varepsilon$ around 0 for ε small enough, and scale

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it so that it extends to a continuous section of $T\mathbb{C}^m|_V$ with an isolated zero at 0. We call $v_{\rm rad}$ *a radial vector field* at $0 \in V$. Notice $v_{\rm rad}$ can be further extended to a tangent vector field $v^{\#}_{\rm rad}$ on U being everywhere transverse to all spheres $\mathbb{S}_\varepsilon$ centered at 0, thus getting a vector field on U which is *radial*. By definition *the Schwartz index* of $v_{\rm rad}$ is the Poincaré–Hopf index at 0 of the radial extension $v^{\#}_{\rm rad}$, so it is $+1$.

Let us consider now a continuous vector field v on V with an isolated singularity at 0. By this we mean a continuous section v of $T\mathbb{C}^m|_V$ which is tangent to $V^* = V \setminus \{0\}$. We want to define *the Schwartz index* of v; this index is related to "the lack of radiality" of the vector field. It has various names in the literature (c.f. [6, 49, 96, 149]), one of them being *radial index*.

We may now define the difference between v and $v_{\rm rad}$ at 0 just as we did in Chap. 1: consider small spheres $\mathbb{S}_\varepsilon$, $\mathbb{S}_{\varepsilon'}$; $\varepsilon > \varepsilon' > 0$, and let w be a vector field on the cylinder X in V bounded by the links $\mathbf{K}_\varepsilon = \mathbb{S}_\varepsilon \cap V$ and $\mathbf{K}_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap V$, such that w has finitely many singularities in the interior of X, it restricts to v on $\mathbf{K}_\varepsilon$ and to $v_{\rm rad}$ on $\mathbf{K}_{\varepsilon'}$. The *difference* of v and $v_{\rm rad}$ is defined as

$$d(v, v_{\rm rad}) = \mathrm{Ind}_{\rm PH}(w, X)\,,$$

the Poincaré–Hopf index of w on X.

Definition 2.1.1 (Schwartz index: case of a variety V with isolated singularity at 0 and v an arbitrary vector field on V with isolated singularity at 0). One defines *the Schwartz index* of v at $0 \in V$ to be:

$$\mathrm{Ind}_{\rm Sch}(v, 0; V) = 1 + d(v, v_{\rm rad})\,.$$

The following result is well-known (see [6, 49, 149]). For vector fields with radial singularities, this is a special case of the work of M.-H. Schwartz; the general case follows easily from this.

Theorem 2.1.1. *Let V be a compact complex analytic variety with isolated singularities $q_1, \cdots, q_r$ in a complex manifold M, and let v be a continuous vector field on V, singular at the q_i and possibly at some other isolated points in V. Let $\mathrm{Ind}_{\rm Sch}(v, V)$ be the sum of the Schwartz indices of v at the q_i plus its Poincaré–Hopf index at the singularities of v in the regular part of V. Then:*

$$\mathrm{Ind}_{\rm Sch}(v, V) = \chi(V)$$

The proof is very simple; we give it here because this illustrates arguments used later. Assume first the vector field v is radial at each $q_1, \cdots, q_r$, so its local Schwartz index at each q_i is 1. Now take small discs D_i in M around each q_i and remove from V the interior of each $V \cap D_i$; we get a manifold V^* which is compact with boundary. The vector field is transverse to the boundary everywhere. Hence its total Poincaré–Hopf index there equals $\chi(V^*)$. The result then follows from the Poincaré–Hopf index theorem because one has:

$$\chi(V) = r + \chi(V^*)\,.$$

Now, if v is nonradial at (some) q_i we do a simple trick: for each $\varepsilon > 0$ sufficiently small, denote by $\mathbb{B}_{i,\varepsilon}$ the ball in M of radius ε around q_i (for some metric) and set $\mathbf{K}_{i,\varepsilon} = V \cap \mathbb{B}_{i,\varepsilon}$ and $V^* = V \setminus \bigcup_i (V \cap \mathbb{B}_{i,\varepsilon})$ for some fixed $\varepsilon > 0$ sufficiently small. By [121] each boundary component $\mathbf{K}_{i,\varepsilon}$ of V^* has a neighborhood diffeomorphic to a cylinder $\mathbf{K}_{i,\varepsilon} \times [0,1]$. Choose $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon > \varepsilon_1 > \varepsilon_2$, and let $X_{\varepsilon,\varepsilon_1}$ and $X_{\varepsilon_1,\varepsilon_2}$ be the cylinders in M bounded by $\{\mathbf{K}_{i,\varepsilon}, \mathbf{K}_{i,\varepsilon_1}\}$ and $\{\mathbf{K}_{i,\varepsilon_1}, \mathbf{K}_{i,\varepsilon_2}\}$ respectively. Put the vector field v on each $\mathbf{K}_{i,\varepsilon_1}$ and on each $\mathbf{K}_{i,\varepsilon}$ and $\mathbf{K}_{i,\varepsilon_2}$ put a radial vector field v_{rad}. Then the local Schwartz index of v at each q_i is 1 plus the difference $d(v, v_{\text{rad}})$ between v on $\mathbf{K}_{i,\varepsilon_1}$ and v_{rad} on $\mathbf{K}_{i,\varepsilon_2}$, which equals $-d(v_{\text{rad}}, v)$, the difference between v_{rad} on $\mathbf{K}_{i,\varepsilon}$ and v on $\mathbf{K}_{i,\varepsilon_1}$. Hence this case reduces to the previous one of radial vector fields, proving theorem 2.1.1. □

The idea for defining the Schwartz index in general, if the singular set has dimension more than 0, is similar in spirit to the case above, but it presents some technical difficulties for which we need to introduce some concepts and notation.

2.2 Whitney Stratifications

Here we make a short summary of what we need in the sequel about stratifications. We refer to [28, 73, 107] for more on the subject. A stratification of a space X is a particularly nice decomposition of this space into pieces, all of which smooth manifolds called the strata.

Definition 2.2.1. Let V be a complex analytic variety of dimension n in some complex manifold M. An *analytic stratification* of V means a locally finite family $(V_\alpha)_{\alpha \in A}$ of nonsingular analytic subspaces of V (*i.e.*, each V_α is a complex manifold) such that:

(1) The family is a partition of V, *i.e.*, they are pairwise disjoint and their union covers V.

(2) For each V_α, the closures in V of both $\overline{V}_\alpha$ and $\overline{V}_\alpha \setminus V_\alpha$ are analytic in V.

(3) For each pair (V_α, V_β) such that $V_\alpha \cap \overline{V}_\beta \neq \emptyset$ one has $V_\alpha \subset \overline{V}_\beta$.

The highest dimensional stratum, which may not be connected, is called *the regular stratum* and usually denoted by V_0 or V_{reg}.

Definition 2.2.2. A stratification $(V_\alpha)_{\alpha \in A}$ of V is said to be Whitney if it further satisfies the following two conditions, known as the Whitney conditions (a) and (b), for every pair (V_α, V_β) such that $V_\alpha \subset \overline{V}_\beta$.

Let $x_i \in V_\beta$ be an arbitrary sequence converging to some point $y \in V_\alpha$ and $y_i \in V_\alpha$ a sequence that also converges to $y \in V_\alpha$. Suppose these sequences are such that (in appropriate Grassmannian) the sequence of secant lines

$l_i = \overline{x_i y_i}$ also converges to some limiting line l, and the tangent planes $T_{x_i}V_\beta$ converges to some limiting plane τ. The Whitney conditions (a) and (b) are the following:

(a) The limit space τ contains the tangent space of the stratum V_α at y, *i.e.*, $T_y V_\alpha \subset \tau$.

(b) The limit space τ contains all the limits of secants, *i.e.*, $l \subset \tau$.

One knows that condition (b) implies condition (a), but it is useful to have both conditions stated explicitly.

Remark 2.2.1. Whitney stratifications are very important for several reasons, some of which will become apparent along this text. Some important facts about these stratifications are:

(1) Every closed (sub)analytic subset of an analytic manifold admits a Whitney stratification.

(2) Whitney stratified spaces can be triangulated compatibly with the stratification.

(3) The transversal intersection of two Whitney stratified spaces is a Whitney stratified space, whose strata are the intersections of the strata of the two spaces.

(4) Whitney stratifications are locally topologically trivial along the strata. That is, given a complex (or real) analytic space V with a Whitney stratification $(V_\alpha)_{\alpha \in A}$, a point $x \in V_\alpha$ and a local embedding of (V, x) in $\mathbb{C}^m$, there is a neighborhood W of x in $\mathbb{C}^m$, diffeomorphic to $\Delta \times U_\alpha$, where U_α is a ball, neighborhood of x in V_α and Δ is a small closed disk through x of complex dimension $m - \dim_{\mathbb{C}} V_\alpha$, transverse to all the strata of V, and such that $W \cap V_\beta = (\Delta \cap V_\beta) \times U_\alpha$ for each stratum V_β with $x \in \overline{V_\beta}$ (see [171, §9], [5]). This is essentially a consequence of the Thom first Isotopy Lemma (see [164]).

2.3 Radial Extension of Vector Fields

Let us describe briefly the radial extension technique developed by M.-H. Schwartz. The idea is simple though there are technical difficulties that we shall omit. A detailed exposition of this construction can be found in Sect. 7 of [33] or in [28].

We consider a complex analytic n-variety, *i.e.*, a reduced complex analytic space V of (complex) dimension n, embedded in a complex manifold M of dimension m and endow M with a Whitney stratification adapted to V; *i.e.*, V is union of strata. Since each stratum V_α is itself a complex manifold we have its tangent bundle TV_α. The singular set of V is denoted by $\mathrm{Sing}(V)$

and the regular one $V_{\mathrm{reg}} = V \setminus \mathrm{Sing}(V)$. If V is reducible, we assume it is pure dimensional.

Definition 2.3.1. A *stratified vector field* on V means a (continuous, smooth) section v of the complex tangent bundle $TM|_V$ such that for each $x \in V$ the vector $v(x)$ is contained in the tangent space of the stratum V_α that contains x.

First we describe the local extension process. It is a consequence of the local topological triviality as explained in iv) of Remark 2.2.1.

Let v_α be a vector field in a neighborhood of a point $x \in V_\alpha$ with possibly a singularity at x. According to iv) of Remark 2.2.1, there is a product neighborhood $W \cong \Delta \times U_\alpha$ of x in the ambient space. We may assume that x is the only one possible singularity of v_α in U_α.

Denoting by $p_1 : W \to \Delta$ and $p_2 : W \to U_\alpha$ the projections on the two factors of the product, we have a decomposition

$$TW = p_1^* T\Delta \oplus p_2^* TU_\alpha.$$

On one hand, the pull-back $p_2^* v_\alpha$ is a vector field on W, which is "parallel" to v_α. It is stratified, since it is tangent to the fibers of p_1. On the other hand, let Δ be equipped with the induced stratification and let v_Δ be a stratified vector field on Δ, which is radial in the usual sense and singular at x. Then $p_1^* v_\Delta$ is a stratified vector field on W since it is tangent to the fibers of p_2 and v_Δ is stratified. It is thus radial in each slice $\Delta \times \{q\}$ for q in U_α. The local radial extension of v_α in W is the following:

Definition 2.3.2. The local radial extension of v_α, denoted by v, is the stratified vector field defined on the neighborhood W as the sum:

$$v = p_1^* v_\Delta + p_2^* v_\alpha.$$

The fundamental property of the local radial extension is the following:

Lemma 2.3.1. *The local radial extension v of v_α has no singularity along the boundary of W and is pointing outward W along its boundary. If v_α has a singularity at x with index* $\mathrm{Ind}_{\mathrm{PH}}(v_\alpha, x; V_\alpha)$*, then the local radial extension v of v_α admits x as unique singular point in W, and one has*

$$\mathrm{Ind}_{\mathrm{PH}}(v, x; W) = \mathrm{Ind}_{\mathrm{PH}}(v_\alpha, x; V_\alpha).$$

Definition 2.3.3 (Schwartz index: case of a stratified variety V and v the local radial extension vector field). Let v be a stratified vector field obtained as in Definition 2.3.2. Then *the Schwartz index* of v at x on V is defined to be the Poincaré–Hopf index of v on W:

$$\mathrm{Ind}_{\mathrm{Sch}}(v, x; V) = \mathrm{Ind}_{\mathrm{PH}}(v, x; W).$$

The local radial extension allows to define the global radial extension. Now we assume V to be compact.

Let us denote the corresponding filtration of V by:

$$V = \overline{V}_{\text{reg}} = \overline{V}_n \supset \overline{V}_{n-2} \supset \cdots \supset \overline{V}_{\alpha_j} \supset \cdots \supset \overline{V}_{\alpha_2} \supset \overline{V}_{\alpha_1} \supset V_{\alpha_0}$$

where V_{α_j} are the (not necessarily connected) strata and V_{α_0} is the lowest dimensional stratum. The radial extension technique is defined by induction on the dimension of the strata.

In the first step, let us consider an arbitrary vector field v_{α_0} with (finitely many) isolated singularities on the stratum V_{α_0}, which is compact since V is. One performs the local radial extension (Definition 2.3.2) in a tube $\mathcal{T}(V_{\alpha_0})$ around V_{α_0} as in (1.1.2) above, union of neighborhoods W as in Definition 2.3.2. Let us denote by v the obtained vector field. If $\dim V_{\alpha_0} = 0$, then v is a stratified radial vector field in a ball around the point $x_0 \in V_{\alpha_0}$ in M. The vector field v is pointing outward $\mathcal{T}(V_{\alpha_0})$ along its boundary and the singularities of v in $\mathcal{T}(V_{\alpha_0})$ are exactly those of v_{α_0} in V_{α_0}. Furthermore, Lemma 2.3.1 implies that the total Poincaré–Hopf index of v on $\mathcal{T}(V_{\alpha_0})$ is $\chi(V_{\alpha_0})$.

The following step is to extend v around V_{α_j} assuming the construction being performed around $V_{\alpha_{j-1}}$. That means v is already constructed in a tube $\mathcal{T}(\overline{V}_{\alpha_{j-1}})$ around $\overline{V}_{\alpha_{j-1}}$, it is pointing outward $\mathcal{T}(\overline{V}_{\alpha_{j-1}})$ along its boundary and if $x \in V_\alpha \subset \overline{V}_{\alpha_{j-1}}$ is a singularity of v, one has $\text{Ind}_{\text{PH}}(v, x; \mathcal{T}(\overline{V}_{\alpha_{j-1}})) = \text{Ind}_{\text{PH}}(v, x; V_\alpha)$.

The vector field v is defined on a neighborhood of $\overline{V}_{\alpha_j} \setminus V_{\alpha_j}$ and can be extended as a vector field still denoted by v with (finitely many) isolated singularities within V_{α_j}. One considers a tube $\mathcal{T}(\overline{V}_{\alpha_j})$ around $\overline{V}_{\alpha_j}$ as in (1.1.2) above, union of the tube $\mathcal{T}(\overline{V}_{\alpha_{j-1}})$ around $\overline{V}_{\alpha_j} \setminus V_{\alpha_j} = \overline{V}_{\alpha_{j-1}}$ and of neighborhoods W (as in Definition 2.3.2) around V_{α_j}. Using the local radial extension property one extends v in $\mathcal{T}(\overline{V}_{\alpha_j})$ in such a way as induction hypotheses are satisfied.

We may summarize the previous discussion in the following theorem of M.-H. Schwartz (see [33] or [28] for a detailed exposition and a complete proof):

Theorem 2.3.1. *([139, 142]) Let V be a complex analytic variety in a complex manifold M, and let $(V_\alpha)_{\alpha \in A}$ be a Whitney stratification of M adapted to V. Then there exists stratified vector fields on a neighborhood of V in M constructed by radial extension as above. Every such vector field v satisfies:*

(1) *Given any stratum (V_α), the total Poincaré–Hopf index of v on $\mathcal{T}(\overline{V}_\alpha)$ is $\chi(\overline{V}_\alpha)$.*

(2) *v is transverse, outwards pointing, to the boundary of every small regular neighborhood of V in M.*

(3) *The Poincaré–Hopf index of v at each singularity x is the same if we regard v as a vector field on the stratum that contains x or as a vector field*

in a neighborhood of x in M. Hence the total Schwartz index of v on V is $\chi(V)$.

2.4 The Schwartz Index on a Stratified Variety

The results described in the previous section tell us that given a compact complex analytic variety V in a complex manifold M which is equipped with a Whitney stratification $\{V_\alpha\}$ adapted to V, we may consider vector fields on V obtained by radial extension. We now make similar considerations for stratified vector fields in general, not necessarily obtained by radial extension.

2.4.1 Case of Vector Fields with an Isolated Singularity

Let v be a stratified vector field on V with isolated singularities. We want to define the Schwartz index of v at these points. Since the question is local, we may assume M is $\mathbb{C}^m$ and the singular point is 0. If 0 is an isolated singularity of V, the Schwartz index of v at 0 is defined in the first section of this chapter. In general this can be done as follows.

Let us denote by V_α the stratum containing 0. We consider two balls $\mathbb{B}_\varepsilon$, $\mathbb{B}_{\varepsilon'}$ centered at 0, with $\varepsilon > \varepsilon' > 0$ without other singularity of v and small enough so that their boundaries are transverse to all strata. Inside the smaller $\mathbb{B}_{\varepsilon'}$ we consider a stratified radial vector field $v_{\rm rad}$ with center 0 and pointing outwards the ball. On the boundary $\partial\mathbb{B}_\varepsilon$ of the larger one, we consider the vector field v. One has

$$\mathrm{Ind}_{\mathrm{PH}}(v_{\rm rad}, 0; M) = +1.$$

Let us consider $K_{\varepsilon,\varepsilon'} = (\mathbb{B}_\varepsilon \setminus \mathrm{Int}\,\mathbb{B}_{\varepsilon'}) \cap V$. On the parts $\mathbb{S}_{\varepsilon'} = \partial\mathbb{B}_{\varepsilon'}$ and $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$ of the boundary of $K_{\varepsilon,\varepsilon'}$ one has a vector field w defined by $v_{\rm rad}$ and v respectively. One extends w in $K_{\varepsilon,\varepsilon'}$ by the radial extension process that we described in the previous section as a stratified vector field with isolated singularities p_j so that we have

$$\mathrm{Ind}_{\mathrm{PH}}(w, p_j; V_\beta) = \mathrm{Ind}_{\mathrm{PH}}(w, p_j; \mathbb{C}^m)$$

where V_β is the stratum containing p_j.

Definition 2.4.1. The *difference* of v and v_{rad} is defined as:

$$d(v, v_{\text{rad}}) = \sum_{\beta} \sum_{j} \text{Ind}_{\text{PH}}(w, p_j; K_{\varepsilon,\varepsilon'}),$$

where the sum on the right runs over the singular points of w in $K_{\varepsilon,\varepsilon'}$. This integer does not depend on the choice of w.

Definition 2.4.2 (Schwartz index: case of a stratified variety V and an arbitrary vector field v with isolated singularity). *The Schwartz index* of v at $0 \in V$ is defined as:

$$\text{Ind}_{\text{Sch}}(v, 0; V) = 1 + d(v, v_{\text{rad}}).$$

It is clear that if V is smooth at 0 then this index coincides with the usual Poincaré–Hopf index; it also coincides with that in Sect. 1 above if 0 is an isolated singularity of V and with the usual index of M.-H. Schwartz (2.3.3), for vector fields obtained by radial extension.

The proof of the following theorem is exactly as that of 2.1.1 and we leave the details to the reader.

Theorem 2.4.1. *Let V be a compact, complex analytic variety in some complex manifold M equipped with a Whitney stratification adapted to V. Let v be a continuous, stratified vector field on a neighborhood of V in M, with isolated singularities $x_1, \cdots, x_s$, all contained in V, and denote by* $\text{Ind}_{\text{Sch}}(v, x_i; V)$ *the corresponding local Schwartz indices at the singular points of V. Then one has:*

$$\chi(V) = \sum_{i=1}^{s} \text{Ind}_{\text{Sch}}(v, x_i; V),$$

where $\chi(V)$ is the Euler–Poincaré characteristic.

An immediate consequence of 2.4.1 is:

Corollary 2.4.1. *Let V be as in Theorem 2.4.1. If there exists a continuous, stratified vector field on a neighborhood of V in M with no singularities, then Euler–Poincaré characteristic of V vanishes:*

$$\chi(V) = 0.$$

2.4.2 Case of Vector Fields with Nonisolated Singularity

Assume now we are given a vector field v defined on the regular part $V_{\text{reg}} = V \setminus \text{Sing}(V)$ of V, and nonsingular away from some subcomplex S_0 of V_{reg}.

This situation was envisaged in [30, 31] (see also [6]). In this context one will associate an index $\mathrm{Ind}(v, S; V)$ to each compact connected component S of $\mathrm{Sing}(V)$; this is the *Schwartz index* of v at S. This construction is relevant for the discussion in Chaps. 8–10 of characteristic classes of singular varieties.

Let S be a connected component of $\mathrm{Sing}(V)$ and $\mathcal{T}(S)$ a cellular tube around S with smooth boundary $\partial\mathcal{T}(S)$ (Remark 1.1.2). Let us denote by $\mathcal{T}_V(S)$ the intersection $\mathcal{T}(S) \cap V$. The boundary $\partial\mathcal{T}(S)$ is transverse to V and intersection $\partial\mathcal{T}(S) \cap V = \partial\mathcal{T}_V(S)$ lies in V_{reg}, hence is smooth.

Let τ be a vector field tangent to V_{reg}, defined in a neighborhood of the boundary $\partial\mathcal{T}_V(S)$ in V_{reg} and pointing outwards $\mathcal{T}_V(S)$ along the boundary.

Then define the Schwartz index of τ at S by:

$$\mathrm{Ind}_{\mathrm{Sch}}(\tau, S) := \chi(\mathcal{T}_V(S)) \; (= \chi(S)).$$

Now, let us consider a vector field v defined and nonsingular on $U \setminus S$ with U a neighborhood of S. We take two cellular tubes $\mathcal{T}(S)$ and $\mathcal{T}'(S)$ as before such that $\mathcal{T}'(S)$ contains the closure of $\mathcal{T}(S)$.

$\mathcal{T}'_V(S)$ of S denoted by N'_S with smooth boundary. For simplicity, we suppose that N'_S contains the closure of N_S. We suppose also that the only singularities of v in N'_S are on S. Let C_S be the cylinder bounded by ∂N_S and $\partial N'_S$, and let ζ be the vector field on ∂C_S which is v on $\partial N'_S$ and τ on ∂N_S. Define the *difference* between v and τ at S, $d(v, \tau)$, just as we did in Chap. 1: it is the total Poincaré–Hopf index of ζ in C_S. Now define the *Schwartz (or radial) index* of v at S to be:

$$\mathrm{Ind}_{\mathrm{Sch}}(v, S) := \mathrm{Ind}_{\mathrm{Sch}}(\tau, S) + d(v, \tau) = \chi(N_S) + d(v, \tau). \tag{2.4.2}$$

One has the following straightforward generalization of the Poincaré–Hopf Theorem.

Theorem 2.4.3. *Let V be a compact, oriented, analytic variety of dimension n. Let $S_1, \ldots, S_p$ be the connected components of the singular set of V. Let v be a continuous vector field on V, singular at $S_1, \ldots, S_p$ and possibly at some isolated points $x_1, \ldots, x_s$ in V_{reg}. Define the total radial index of v in V, $\mathrm{Ind}_{\mathrm{Sch}}(v, V)$, to be the sum of the radial indices $\mathrm{Ind}_{\mathrm{Sch}}(v, S_\lambda)$ at $S_1, \ldots, S_p$ and the usual Poincaré–Hopf index at $x_1, \ldots, x_s$. Then*

$$\mathrm{Ind}_{\mathrm{Sch}}(v, V) = \chi(V),$$

independently of v.

Proof. Let S_λ , $\lambda = 1, \ldots, p$, be as above. Near each S_λ, the radial vector field v_{rad} is transverse to the smooth boundary K_λ of a regular neighborhood N_λ of S_λ. Therefore

$$\mathrm{Ind}_{\mathrm{Sch}}(v_{\mathrm{rad}}, S_\lambda) = \chi(S_\lambda),$$

by definition. Let $\overset{\circ}{N}_\lambda$ be the interior of N_λ, a regular neighborhood of S_λ, and set $V^* = V - \{\overset{\circ}{N}_1 \cup ... \cup \overset{\circ}{N}_p\}$. Then V^* is a compact manifold with boundary $K = \{K_1 \cup ... \cup K_p\}$, and the vector field $v_{\rm rad}$ is transverse to K and points inwards. Therefore $\operatorname{Ind}_{\rm PH}(v_{\rm rad}, V^*) = \chi(V^*) - \chi(K)$, by the theorem of Poincaré–Hopf for manifolds with boundary (I.1.2 above). On the other hand

$$\chi(V) = \chi(V^*) + \{\chi(S_1) + \cdots + \chi(S_p)\} - \chi(K).$$

Thus

$$\chi(V) = \operatorname{Ind}_{\rm PH}(v_{\rm rad}, V^*) + \{\operatorname{Ind}_{\rm Sch}(v_{\rm rad}, S_1) + \cdots + \operatorname{Ind}_{\rm Sch}(v_{\rm rad}, S_p)\},$$

hence $\chi(V) = \operatorname{Ind}_{\rm Sch}(v_{\rm rad}, V)$.

Now let v be some other vector field on V, singular at $S_1, \ldots, S_p$ and possibly at some smooth points of V. Then by definition:

$$\operatorname{Ind}_{\rm Sch}(v, S_\lambda) = \chi(S_\lambda) + d_\lambda(v, v_{\rm rad})\,,$$

for each λ, where $d_\lambda(v, v_{\rm rad})$ is the difference introduced before. Similarly, the Poincaré–Hopf index of v in V^* is

$$\operatorname{Ind}_{\rm PH}(v, V^*) = \chi(V^*) - \chi(K) + \{d_1(v, v_{\rm rad}) + \cdots + d_r(v, v_{\rm rad})\}\,.$$

Hence

$$\chi(V) = \operatorname{Ind}_{\rm PH}(v, V^*) + \{\operatorname{Ind}_{\rm Sch}(v, S_1) + \cdots + \operatorname{Ind}_{\rm Sch}(v, S_r)\} = \operatorname{Ind}_{\rm Sch}(v, V)\,,$$

as claimed, because $d_\lambda(v, v_{\rm rad}) + d_\lambda(v_{\rm rad}, v) = 0$.

Remark 2.4.1. We notice that one has:

(1) $\operatorname{Ind}_{\rm Sch}(v_{\rm rad}, S) = \chi(S)$ for radial vector fields,

(2) the radial index coincides with the Poincaré–Hopf index if $S \subset V_{\rm reg}$ and

(3) this index is independent of the ambient manifold M.

Given a singular variety V in a complex manifold M as above, and a stratified vector field v on a neighborhood $\widehat{U} \subset M$ of a connected component $S \subset \operatorname{Sing}(V)$ whose singularities are all in S, we have defined above two types of Schwartz indices: on one hand, at each singularity x_i of v we attach an index $\operatorname{Ind}_{\rm Sch}(v, x_i)$ and we have the sum over all of them. On the other hand we can forget we have v on S, take it only on $M \setminus S$ and define an index $\operatorname{Ind}_{\rm Sch}(v, S)$ as above. Here we determine the relation among these indices. Using the above construction, we have:

Theorem 2.4.4. *Let S be a connected component of the singular set of V. We equip M with a Whitney stratification adapted to V and S. Let v be a stratified vector field in a neighborhood $\widehat{U}$ of S in M without singularity on $\partial\widehat{U}$ and whose singular points x_i in $\widehat{U}$ are all contained in S. Let $\mathrm{Ind}_{\mathrm{Sch}}(v, S; V)$ be the index of v at S as defined in 2.4.2, and at each singular point $x_1, \cdots, x_s$ of v on S let $\mathrm{Ind}_{\mathrm{Sch}}(v, x_i; V)$ be the local index defined in 2.4.2. Then one has:*

$$\mathrm{Ind}_{\mathrm{Sch}}(v, S; V) = \sum_{i=1}^{s} \mathrm{Ind}_{\mathrm{Sch}}(v, x_i; V).$$

Chapter 3
The GSV Index

Abstract One of the basic properties of the local Poincaré–Hopf index is stability under perturbations. In other words, if a vector field has an isolated singularity on an open set in $\mathbb{R}^n$ and if we perturb it slightly, then the singularity may split into several singular points, with the property that the sum of the indices of the perturbed vector field at these singular points equals the index of the original vector field at its singularity. If we now consider an analytic variety V defined by a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at 0, and if v is a vector field on V, with an isolated singularity at 0, then one may like "the index" of v at 0 to be stable under small perturbations of both, the function f and the vector field v. This leads naturally to another concept of index, called the GSV index, introduced by X. Gómez-Mont, J. Seade and A. Verjovsky in [71, 144] for hypersurface germs, and extended in [147] (see also [149]) to complete intersections. In this chapter we define this index and we study some of its basic properties. We first do it when the ambient space is an isolated complete intersection singularity (ICIS for short), then we explain the recent generalization in [34] to the case where the ambient variety has nonisolated singularities; this relies on a proportionality theorem similar to the one proved in [33] for the local Euler obstruction, that is discussed later in the text.

In the following chapters we will study other related indices: the GSV index can be interpreted via Chern–Weil theory as the *virtual index* introduced by D. Lehmann, M. Soares and T. Suwa in [111], that we study in Chap. 5. And if the vector field v is holomorphic, then the GSV index also coincides with the *homological index* of Gómez-Mont [68], that we describe in Chap. 7. There is also a recently defined *logarithmic index* in [7], which coincides with the homological index and therefore, for ICIS, with the GSV index.

3.1 Vector Fields Tangent to a Hypersurface

The index we discuss in this chapter is associated to vector fields on germs of hypersurface (or, more generally, complete intersection) singularities, and

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the way how these vector fields extend to the ambient space. So we begin with a brief discussion of this topic.

Let us denote by $(V, 0)$ the germ of a complex analytic hypersurface in $\mathbb{C}^{n+1}$ given by a holomorphic function

$$f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0),$$

which is defined on a small ball $\mathbb{B}_\varepsilon$ and has a unique critical point at 0. Let v be a continuous section of the bundle $T\mathbb{C}^{n+1}|_V$. We notice that for each $x \in V^* = V \setminus \{0\}$, the tangent space $T_x V^*$ consists of all vectors in $T_x\mathbb{C}^{n+1}$ which are mapped to 0 by the derivative of f:

$$T_x V^* = \{\zeta \in T_x\mathbb{C}^{n+1} \,\big|\, df_x(\zeta) = 0\}.$$

For example, if f is the polynomial map in $\mathbb{C}^2$ defined by $f(z_1, z_2) = z_1^2 + z_2^3$, then the line tangent to $V = f^{-1}(0)$ at a point (z_1, z_2), other than the origin, is spanned by the vector $\widetilde{\zeta}(z_1, z_2) = (-3z_2^2, 2z_1)$. To see this notice one has

$$df_z = 2\, z_1 dz_1 + 3\, z_2^3 dz_2.$$

Hence:

$$df_z(\widetilde{\zeta}) = df_z(-3z_2^2, 2z_1) = 0.$$

Now, a vector field v on V can be thought of as being a continuous map $(V, 0) \xrightarrow{v} (\mathbb{C}^{n+1}, 0)$ which is nonzero on V^* and whose image is contained in the linear space tangent to V at each given point. Since V is a closed subset of $\mathbb{B}_\varepsilon$, this map extends to a neighborhood of V in $\mathbb{C}^{n+1}$. Geometrically this means that the vector field v on V can always be extended to the ambient space, or equivalently that v can always be considered as the restriction to V of a vector field in the ambient space. However the extension of v to V is by no means unique. Furthermore, all these statements also hold in the holomorphic category:

Theorem 3.1.1. ([16]) *Let V be a complex analytic variety in $\mathbb{C}^m$ with an isolated singularity at 0. Then:*

(1) *There exist holomorphic vector fields on V with an isolated singularity at 0. In fact the space of such vector fields is infinite-dimensional.*

(2) *If v is a holomorphic vector field on V with an isolated singularity, then there are infinitely many extensions of v to a neighborhood of 0 in the ambient space with an isolated singularity.*

As an example, if V is defined in $\mathbb{C}^2$ by a map $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, then the Hamiltonian vector field $\widetilde{\zeta}(z_1, z_2) = (-\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_1})$ is tangent to V and it is zero only at the origin. Notice that this vector field is actually tangent to all the fibers $f^{-1}(t)$. Let ζ be the restriction of $\widetilde{\zeta}$ to V. Notice that ζ can be extended to $\mathbb{C}^2$ in many other ways; for example, if g is a holomorphic function on $\mathbb{C}^2$ that vanishes exactly on V and represents a nonzero element in the local ring $\mathcal{O}_{(\mathbb{C}^2,0)}$, then

$$\xi = \left(g - \frac{\partial f}{\partial z_2}, g + \frac{\partial f}{\partial z_1}\right),$$

coincides with ζ on V and is no longer tangent to the fibers of f; choosing g appropriately we can also assure that ξ has an isolated singularity at 0.

In $\mathbb{C}^3$ one has the following example of [68] (see also 3.3.2 and §2 in Chap. 7). Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ have an isolated critical point at 0, set $V = f^{-1}(0)$ and choose the coordinates (z_1, z_2, z_3) so that V meets only at 0 the analytic set where the partial derivatives of f with respect to z_2 and z_3 vanish:

$$V \cap \left\{ \frac{\partial f}{\partial z_2} = \frac{\partial f}{\partial z_3} = 0 \right\} = \{0\}.$$

Define a holomorphic vector field in $\mathbb{C}^3$ by

$$\widetilde{\zeta} = \left(f, \frac{\partial f}{\partial z_3}, -\frac{\partial f}{\partial z_2}\right),$$

Notice $\widetilde{\zeta}$ has an isolated singularity at 0 and

$$df(\widetilde{\zeta}) = f \frac{\partial f}{\partial z_1},$$

hence $df(\widetilde{\zeta})$ vanishes at the points where f vanishes, so $\widetilde{\zeta}$ is tangent to V. If we set $\zeta = \widetilde{\zeta}|_V$, then we have a holomorphic vector field on V with an isolated singularity at the origin, and an extension $\widetilde{\zeta}$ of it to $\mathbb{C}^3$ which also has an isolated singularity. Notice however that, unlike the previous example, $\widetilde{\zeta}$ is no longer tangent to the fibers of f. Yet, we may forget we are given $\widetilde{\zeta}$ and just consider the vector field ζ on V. Since f vanishes exactly on V, ζ takes the form $\zeta = (0, \frac{\partial f}{\partial z_3}, -\frac{\partial f}{\partial z_2})$ and we can extend it to a holomorphic vector field $\widetilde{\xi}$ on $\mathbb{C}^3$ defined by:

$$\widetilde{\xi} = \left(0, \frac{\partial f}{\partial z_3}, -\frac{\partial f}{\partial z_2}\right).$$

This is tangent to all the nonsingular hypersurfaces $f^{-1}(t)$, $t \neq 0$. The singular set of $\widetilde{\xi}$ is the complete intersection curve defined by the ideal $(\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3})$, which meets each nonsingular fiber $f^{-1}(t)$ at finitely many points, whose total sum (counting multiplicities) is constant (see Chap. 7). This constant is an index that depends only on ζ and the way V is embedded in $\mathbb{C}^3$. This is the index that we study below.

Remark 3.1.1. We restricted the above discussion to vector fields on hypersurfaces for simplicity, as an example. Of course it is important to consider vector fields "tangent" to a singular variety in general. This is done for instance in [38] and [65]. In [38] J. Bruce and R. Roberts show that given the germ $(V, 0)$ of a complex analytic variety with a possibly nonisolated singularity at 0, there is a canonical stratification of V, that they call *logarithmic*, for which each holomorphic tangent vector field is stratified, and furthermore the tangent space of each stratum is generated by such vector fields. In general, the logarithmic stratification may not be locally finite; when it is locally finite, then it is also Whitney.

3.2 The Index for Vector Fields on ICIS

Let $(V, 0)$ be a germ of complex analytic variety of dimension n in $\mathbb{C}^{n+k}$. Recall that (cf. [116, (1.5)]) $(V, 0)$ is a *geometric complete intersection*, if it is defined as the common zero set of (germs of) k holomorphic functions. Also, $(V, 0)$ is a *complete intersection* if the ideal $\mathcal{I}_V$ in $\mathcal{O}_{n+k}$ of function vanishing on V is generated by k holomorphic functions.

Let $(V, 0)$ be a germ of a complete intersection of dimension n with an isolated singularity, defined by a holomorphic map

$$f = (f_1, \dots, f_k) : (\mathbb{C}^{n+k}, 0) \longrightarrow (\mathbb{C}^k, 0),$$

i.e., $f_1, \dots, f_k$ generate the ideal $\mathcal{I}_V$. If $n = 1$ we further assume (for the moment, cf. Remark 3.2.2) that V is irreducible. Just as in [116], we abbreviate an isolated complete intersection singularity germ as ICIS.

Since 0 is an isolated singularity of V, it follows that the (complex conjugate) gradient vector fields $\{\overline{\text{grad}} f_1, \dots, \overline{\text{grad}} f_k\}$ are linearly independent everywhere away from 0 and they are normal to V (for the usual hermitian metric in $\mathbb{C}^{n+k}$). Let v be a continuous vector field on V singular only at 0. The set $\{v(x), \overline{\text{grad}} f_1(x), \dots, \overline{\text{grad}} f_k(x)\}$ is a $(k + 1)$-frame at each point in $V^* := V \setminus \{0\}$, and up to homotopy, it can be assumed to be orthonormal, *i.e.*, each vector has norm 1 and they are pairwise orthogonal. Thence these vector fields define a continuous map from V^* into the Stiefel manifold of complex orthonormal $(k + 1)$-frames in $\mathbb{C}^{n+k}$, denoted $W_{k+1}(n + k)$.

Let $\mathbf{K} = V \cap \mathbb{S}_\varepsilon$ be the link of 0 in V. It is an oriented, real manifold of dimension $(2n - 1)$ and the above frame defines a continuous map

$$\phi_v = (v, \overline{\text{grad}} f_1, \dots, \overline{\text{grad}} f_k) : \mathbf{K} \longrightarrow W_{k+1}(n + k).$$

The Stiefel manifold $W_{k+1}(n + k)$ is diffeomorphic to the homogeneous space $U(n + k)/U(n - 1)$ and therefore the homotopy sequence associated to this fibration implies that $W_{k+1}(n + k)$ is $(2n - 2)$-connected, while its homology in dimension $(2n-1)$ is isomorphic to $\mathbb{Z}$. Hence the map ϕ_v has a well defined degree $\deg(\phi_v) \in \mathbb{Z}$, defined by means of the induced homomorphism $H_{2n-1}(\mathbf{K}) \to H_{2n-1}(W_{k+1}(n+k))$ in the usual way. For this we consider the generators of each of these groups corresponding to the element [1]. Notice that $W_{k+1}(n+k)$ is a fiber bundle over $W_k(n+k)$ with fiber the sphere $\mathbb{S}^{2n-1}$; if $(e_1, \cdots, e_{n+k})$ is the canonical basis of $\mathbb{C}^{n+k}$, then the fiber γ over the k-frame $(e_1, \cdots, e_k)$ determines the canonical generator $[\gamma]$ of $H_{2n-1}(W_{k+1}(n+k)) \simeq \mathbb{Z}$. If $[\mathbf{K}]$ is the fundamental class of $\mathbf{K}$, then $(\phi_v)_*[\mathbf{K}] = \lambda \cdot [\gamma]$ for some integer λ and the degree of ϕ_v is given by:

$$\deg(\phi_v) = \lambda.$$

Alternatively one can prove that every map from a closed oriented $(2n-1)$-manifold into $W_{k+1}(n+k)$ factors by a map into the fiber $\gamma \simeq \mathbb{S}^{2n-1}$, essentially by transversality. Hence ϕ_v represents an element in $\pi_{2n-1} W_{k+1}(n+k) \simeq \mathbb{Z}$, so ϕ_v is classified by its degree. In other words, up to homotopy, the map ϕ_v can be regarded as a map from the link $\mathbf{K}$ into the sphere $\mathbb{S}^{2n-1}$, and $\deg(\phi_v)$ is its degree in the usual sense (c.f. the following chapter where this discussion is carefully done in the real case, which is more delicate).

Definition 3.2.1. The GSV index of v at $0 \in V$, $\mathrm{Ind}_{\mathrm{GSV}}(v, 0)$, is the degree of the above map ϕ_v.

This index depends not only on the topology of V near 0, but also on the way V is embedded in the ambient space. For example, the singularities in $\mathbb{C}^3$ defined by

$$\{x^2 + y^7 + z^{14} = 0\} \text{ and } \{x^3 + y^4 + z^{12} = 0\},$$

are orientation preserving homeomorphic as abstract varieties, disregarding the embedding, and one can prove that the GSV index of the radial vector field is 79 in the first case and 67 in the latter; this follows from the fact (see 3.2.1 below) that for radial vector fields the GSV index is $1 + (-1)^{\dim V} \mu$, where μ is the Milnor number, which in the examples above is known to be 78 and 66 respectively, by [121, Theorem 9.1].

We recall that one has a Milnor fibration associated to the map f, see [79, 116, 121], and the Milnor fiber $\mathbf{F}$ can be regarded as a compact $2n$-manifold with boundary $\partial \mathbf{F} = \mathbf{K}$. Moreover, by the Transversal Isotopy Lemma there is an ambient isotopy of the sphere $\mathbb{S}_\varepsilon$ taking $\mathbf{K}$ into $\partial \mathbf{F}$, which can be extended to a collar of $\mathbf{K}$, which goes into a collar of $\partial \mathbf{F}$ in $\mathbf{F}$. Hence v can be regarded as a nonsingular vector field on $\partial \mathbf{F}$.

Theorem 3.2.1. *The GSV index has the following properties:*

(1) *The GSV index of v at 0 equals the Poincaré–Hopf index of v in the Milnor fiber:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v, 0) = \mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}).$$

(2) *If v is everywhere transverse to* $\mathbf{K}$, *then*

$$\mathrm{Ind}_{\mathrm{GSV}}(v,0) = 1 + (-1)^n \mu\,,$$

where n is the complex dimension of V and μ is the Milnor number of 0.

(3) *One has:*

$$\mu \;=\; (-1)^n \big(\mathrm{Ind}_{\mathrm{GSV}}(v,0) \;-\; \mathrm{Ind}_{\mathrm{Sch}}(v,0)\big),$$

independently of the choice of v.

Proof. Since the germ $(V,0)$ is an ICIS, the conjugate gradient vector fields $\{\overline{\mathrm{grad}} f_1, \ldots, \overline{\mathrm{grad}} f_k\}$ are all linearly independent everywhere on $\mathbf{F}$ and normal to this manifold, so the degree of ϕ_v can be identified with the obstruction to extending v to a tangent vector field on $\mathbf{F}$. Hence $\mathrm{Ind}_{\mathrm{GSV}}(v,0) = \mathrm{Ind}_{\mathrm{PH}}(v,\mathbf{F})$ as claimed in (1). Statement (2) follows from statement (1) together with Theorem 1.1.2 above and the fact that, by [79, 116, 121], the Euler–Poincaré characteristic of $\mathbf{F}$ is $1 + (-1)^n \mu$. For (3) we first notice that this statement follows from (2) if v is radial. The general case follows from this together with 1.1.2.

Remark 3.2.1. Theorem 3.2.1 says that if we perturb the mapping f by adding to it a small constant, then *the index is preserved* in the sense that the GSV index of the vector field on the singular fiber becomes the sum of Poincaré–Hopf indices in the nearby fibers. One may of course look at more general deformations of the map-germ f. From the previous discussion we see that the way the GSV index changes when we perturb f does not depend on the choice of vector field, but only on the way the topology of the Milnor fiber changes, *i.e.*, on the behavior of the Milnor number under perturbations. This is an interesting subject that has been studied by several authors, including Lazzeri, Gabrielov, Lê and Massey, among many others.

One has a Poincaré–Hopf type theorem for this index:

Theorem 3.2.2. *Let V be a compact, complex analytic variety with isolated singularities $x_1, \ldots, x_r$, which are all isolated complete intersection germs. Let v be a continuous vector field on V, singular at the x_i's and possibly at some other smooth points $y_1, \ldots, y_s$ of V. Let* $\mathrm{Ind}_{\mathrm{GSV}}(v,V)$ *denote the total GSV index of v, i.e., the sum of the local GSV indices at the x_i's and the usual Poincaré–Hopf indices at the y_i's. Then one has:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v,V) = \chi(V) + (-1)^n \sum_{i=1}^{r} \mu(x_i),$$

where $\mu(x_i)$ is the Milnor number of V at x_i.

Proof. The proof of this theorem is very similar to that of 2.1.1. One removes from V conical neighborhoods N_i of the x_i and replaces these by copies

$\mathbf{F}_1, \dots, \mathbf{F}_r$ of the corresponding Milnor fibers. The new manifold $\widehat{V}$ is smooth, and if v is radial at each x_i then its Euler characteristic is $\mathrm{Ind}_{\mathrm{GSV}}(v, V)$, essentially by definition. But $\chi(\widehat{V})$ equals $\chi(V) + (-1)^n \sum_{i=1}^r \mu(x_i)$, proving the theorem when v is radial at each x_i. The general case follows easily from this and Proposition 1.1.2.

Remark 3.2.2. In the above discussion we ruled out the case where the dimension of V is 1 and V has several branches. This case is of course interesting and it was first considered by Brunella [39, 40] and Khanedani–Suwa [93] in their study of holomorphic 1-dimensional foliations on complex surfaces (cf. Chap. 6). In this case the GSV index is defined as the Poincaré–Hopf index of an extension of v to a Milnor fiber. If a curve C has only one branch at a singular point x_0 this coincides with Definition 3.2.1. But if C has several branches at x_0 one has an integer attached by 3.2.1 to each branch. If C is a plane curve, the relation among all these indices is well understood and it is determined by the intersection number of the various branches. In fact, Milnor in [121, Theorem 10.5 and Remark 10.10] proved that if $C_1, \dots, C_r$ are the irreducible components of C then one has the formula:

$$\mu = \sum_{i=1}^{r} \mu_i + 2I - r + 1,$$

where μ, respectively μ_i, is the Milnor number of C, respectively C_i, at x_0 and I is defined as $\sum_{i<j} C_i \cdot C_j$. This formula implies that if v is a vector field on C then we have

$$\mathrm{Ind}_{\mathrm{GSV}}(v, x_0; C) := \sum_{i=1}^{r} \mathrm{Ind}_{\mathrm{GSV}}(v, x_0; C_i) = \mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t) - 2I,$$

where $\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t)$ is the Poincaré–Hopf index of an extension of v to a Milnor fiber $\mathbf{F}_t$ of C at x_0, a formula proved independently in [39] and [93].

Remark 3.2.3. We notice that the definition of the GSV index works equally well for singularities which are only geometric complete intersections [116], not necessarily algebraic complete intersections, *i.e.*, all we need is that the gradient vector fields of the functions that define V are linearly independent everywhere on $V \setminus \{0\}$.

3.3 Some Applications and Examples

Example 3.3.1. In Sect. 1 we saw how given a function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at 0, the Hamiltonian vector field $(-\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_1})$ is tangent to $V = f^{-1}(0)$ and to all the fibers of f. Hence its GSV index is zero. More generally, consider a holomorphic function $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}, 0)$, with an isolated critical point at 0. Its differential is

$$df = \frac{\partial f}{\partial z_1} dz_1 + \cdots + \frac{\partial f}{\partial z_{2n}} dz_{2n}.$$

Let $\widetilde{\zeta}$ be an arbitrary vector field obtained by permuting in pairs the components of Df and changing the sign in one of the components in each pair, e.g.

$$\widetilde{\zeta} = \Big(-\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_1}, -\frac{\partial f}{\partial z_4}, \frac{\partial f}{\partial z_3} \Big)$$

or

$$\widetilde{\zeta} = \Big(-\frac{\partial f}{\partial z_3}, -\frac{\partial f}{\partial z_4}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2} \Big).$$

In all cases one has $df(\widetilde{\zeta}) \equiv 0$ everywhere. This means that all these vector fields are tangent to all fibers of f, in particular to $V = f^{-1}(0)$, so its GSV index on V is zero.

Example 3.3.2. The following example is taken from [68] and will play an important role in Chap. 7 for identifying the GSV and the homological indices. Denote the coordinates of $\mathbb{C}^{2n+1}$ by $(z_0, z_1, \cdots, z_{2n})$ and, given a holomorphic function $f : (\mathbb{C}^{2n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at 0, consider the vector field:

$$\widetilde{\zeta} = \Big(f, \frac{\partial f}{\partial z_2}, -\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{2n}}, -\frac{\partial f}{\partial z_{2n-1}} \Big).$$

Assume we have chosen the coordinates in such a way that $\widetilde{\zeta}$ has an isolated singularity at 0, *i.e.*, the hypersurface $V = \{f = 0\}$ meets only at 0 the set $\bigcap_{i=1}^{2n} \{ \frac{\partial f}{\partial z_i} = 0\}$. We set $\zeta = \widetilde{\zeta}|_V$. Notice one has:

$$df(\widetilde{\zeta}) = f \frac{\partial f}{\partial z_0},$$

hence ζ is tangent to V and is also restriction to V of the vector field:

$$\widetilde{\xi} = \Big(0, \frac{\partial f}{\partial z_2}, -\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{2n}}, -\frac{\partial f}{\partial z_{2n-1}} \Big),$$

defined in the ambient space and which is tangent to all the nonsingular hypersurfaces $f^{-1}(t)$, $t \neq 0$. The singular set of $\widetilde{\xi}$ is the complete intersection curve defined by the ideal $(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{2n}})$, which meets each nonsingular fiber $f^{-1}(t)$ at finitely many points, whose total sum (counting multiplicities) is the GSV index of ζ on V. A direct computation then shows that this index, being the intersection number of two complex varieties is equal to:

$$\mathrm{Ind}_{\mathrm{GSV}}(\zeta, 0; V) \;=\; \dim_{\mathbb{C}}\, \mathcal{O}_{2n+1}\Big/\Big(f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{2n}}\Big),$$

this is the dimension of the complex vector space of germs of holomorphic functions at 0 divided by the ideal $(f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{2n}})$:

Example 3.3.3. Consider now a *Pham–Brieskorn* polynomial in $\mathbb{C}^{n+1}$ with coordinates $(z_1, \ldots, z_{n+1})$, that is a polynomial:

$$f(z) = z_1^{d_1} + \cdots + z_{n+1}^{d_{n+1}},$$

where d_i are integers ≥ 2. The variety

$$V = f^{-1}(0) = \{z_1^{d_1} + \cdots + z_{n+1}^{d_{n+1}} = 0\}$$

is a hypersurface (complex codimension 1) with an isolated singularity at $0 \in \mathbb{C}^{n+1}$.

Let d be the least common multiple of the d_i, $i = 1, \ldots, n+1$, set $q_i = d/d_i$ and consider the action of the nonzero complex numbers $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by:

$$t \cdot (z_1, \ldots, z_{n+1}) = (t^{q_1} z_1, \ldots, t^{q_{n+1}} z_{n+1}).$$

For each $t \in \mathbb{C}^*$ one has:

$$f(t \cdot (z_1, \ldots, z_{n+1})) = f(t^{q_1} z_1, \ldots, t^{q_{n+1}} z_{n+1}) = t^d \cdot f(z_1, \ldots, z_{n+1}).$$

Hence V is an invariant set for this action, *i.e.*, V is a union of $\mathbb{C}^*$-orbits. Every holomorphic flow determines a holomorphic vector field, and it is an exercise to see that the vector field v_{rad} on V corresponding to the above $\mathbb{C}^*$-action on $V \setminus \{0\}$ is radial. If all d_i are equal, so that V is homogeneous, then the vector field v_{rad} is the usual radial vector field $(z_1, \ldots, z_{n+1})$ up to a constant. Thus, by 3.2.1 one has:

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0; V) \;=\; 1 + (-1)^n\, \mu(V),$$

where $\mu(V)$ is the Milnor number of V. From [121], see also Example 5.7.1 below, we know:

$$\mu(V) \;=\; (d_1 - 1)(d_2 - 1) \cdots (d_{n+1} - 1).$$

Hence:

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0; V) \;=\; 1 + (-1)^n [(d_1 - 1)(d_2 - 1) \cdots (d_{n+1} - 1)]\,.$$

Thus we conclude:

(1) If n is even, then $\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0; V) > 0$, so every extension of v_{rad} to a continuous vector field in the ambient space, tangent to some fibers of f (or to all of them) will have singularities in each fiber where it is tangent to the fiber.

(2) If n is odd and at least one d_i is ≥ 3, then $\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0; V) < 0$. This means that v_{rad} cannot be extended to a holomorphic vector field in the ambient space being tangent to a fiber of f and having isolated singularities on this fiber. If we could do that, then the index would be positive.

(3) If n is odd and all d_i are 2, so that $\mu(V) = 1$, then $\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0; V) = 0$ and we can extend v_{rad} to the ambient space being tangent to the fibers of f, and nonsingular there. The way to do this extension is not evident at a first glance, but this is actually easy: suppose for simplicity that $n = 1$, so we are in $\mathbb{C}^2$ (the argument in general is an obvious extension of this one). Write the function f as

$$f(z_1, z_2) = z_1^2 + z_2^2,$$

so that $v_{\mathrm{rad}} = (z_1, z_2)$ (we may drop the constant 2). Then consider the family of vector fields $v_t = (1-t)\, v_{\mathrm{rad}} + t\,\zeta$, where t takes values in the interval $[0, 1]$ and ζ is the Hamiltonian vector field $\zeta(z_1, z_2) = (-z_2, z_1)$. For $t = 0$ we have v_{rad}, for $t = 1$ this is ζ, and for each $t \in (0, 1)$ this is a vector field tangent to V and with an isolated singularity at 0. This family allows us to deform the radial vector field v_{rad} (on V) continuously into the vector field $(-z_2, z_1)$, which extends in the obvious way to the ambient space, being singular to all fibers of f and having a unique zero at 0.

3.4 The Case of Isolated Smoothable Singularities

More generally, let 0 be an isolated singularity in a pure dimensional complex analytic variety Y. We say that this singularity is *smoothable* if there exists a complex analytic variety X and a nonconstant analytic map,

$$F \colon X \longrightarrow \mathbb{C},$$

such that $F^{-1}(0)$ is (isomorphic to) Y and $F^{-1}(t)$ is nonsingular for all t near 0. Assume for simplicity that X is embedded in an open subset U of $\mathbb{C}^m$. We know from [102, Th. 1.1] that for every $\varepsilon >> \eta > 0$ sufficiently small, the restriction

$$F : F^{-1}(\mathbb{S}_\eta) \cap \mathbb{B}_\varepsilon \to \mathbb{S}_\eta,$$

is a fiber bundle over $\mathbb{S}_\eta = \{z \in \mathbb{C} \,\big|\, |z| = \eta\}$. Therefore $\chi(\mathbf{F}_t)$, *the Euler–Poincaré characteristic* of each fiber

$$\mathbf{F}_t = F^{-1}(t) \cap \mathbb{B}_\varepsilon, \quad t \in \mathbb{S}_\eta,$$

is independent of t.

Now denote by v a continuous vector field on Y with an isolated singularity at 0. By the Transversal Isotopy Lemma (see [4]) the intersection of Y with the boundary sphere $\mathbb{S}_\varepsilon$ of $\mathbb{B}_\varepsilon$ is isotopic to the intersection of $F^{-1}(t)$ with this sphere. It follows that we can think of v as being a vector field around the boundary $\partial\mathbf{F}_t$ of $\mathbf{F}_t$. By Theorem 1.1.2 above we know that we can extend v to the interior of $\mathbf{F}_t$ with a finite number of singularities, each of which has its local Poincaré–Hopf index; the sum of all these local indices is the total Poincaré–Hopf index of v in $\mathbf{F}_t$, that we denote by $\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t)$. One has:

Proposition 3.4.1. *The number* $\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t)$ *is independent of the choice of* t *and of the extension of* v *to the interior of this fiber. In particular, if* v *is everywhere transverse to the local link of* 0 *in* Y*, then one has:*

$$\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t) = \chi(\mathbf{F}_t).$$

Proof. If v is everywhere transverse to the local link of 0 in Y, which is isotopic to the boundary of the fibers $\mathbf{F}_t$, the result is an immediate consequence of the theorem of Poincaré–Hopf for manifolds with boundary (1.1.2), together with the fact [102] that the $\mathbf{F}_t{}'s$ are the fibers of a fiber bundle. The proof in general follows from this theorem together with 1.1.2.

Definition 3.4.1. We define *the GSV index of* v in Y *relative to the smoothing* given by F by:

$$\mathrm{Ind}_{\mathrm{GSV}}(v, Y; F) = \mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t).$$

It is worth noting that this index does depend on the choice of the analytic map F chosen as a smoothing of Y and not only on Y and X. However, it is shown in [29] that if the smoothing is given by a general linear form, then this index determines the local Euler obstruction of X at 0 (see Chap. 8), which depends only on Y. Hence the GSV index is independent of the smoothing if this is given by a general linear form (this can also be proved directly). We also notice that if $(Y, 0)$ is a complete intersection germ then the smoothing is essentially unique, because the base space of the universal deformation is connected, and as we know already one has:

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, Y; F) = 1 + (-1)^n \mu,$$

where n is the dimension of Y and μ is its Milnor number at 0.

3.5 Nonisolated Singularities

The results in this section are proved in [34]. Here we extend the notion of GSV index to vector fields on complete intersection germs with nonisolated singularities, so long as one has the strict Thom w_f-condition. We begin by recalling this condition. Then we define the index and prove the proportionality theorem of [34] for this index using a geometric argument.

3.5.1 The Strict Thom Condition for Complex Analytic Maps

Let $f : X \to Y$ be a morphism of reduced complex analytic spaces.

Definition 3.5.1. The morphism f is *stratifiable* if there exist Whitney stratifications $(V_\alpha)_{\alpha \in A}$, $(W_\gamma)_{\gamma \in B}$ of X and Y respectively, such that for each $\alpha \in A$ there exists $\gamma(\alpha) \in B$ for which the restriction $f_\alpha = f|_{V_\alpha} : V_\alpha \to W_{\gamma(\alpha)}$ is a complex analytic submersion.

By [82, Sect. 3], every proper complex analytic morphism is stratifiable. In this case the first Isotopy Theorem of Thom–Mather implies that for each stratum W_γ the induced morphism

$$f|_{f^{-1}(W_\gamma)} : f^{-1}(W_\gamma) \longrightarrow W_\gamma$$

is a topologically trivial fibration (see [106]).

Definition 3.5.2. Given $\alpha, \beta \in A$ with $V_\alpha \subset \overline{V}_\beta$, one says that f satisfies *the Thom condition* for the pair (V_β, V_α) at a point $x \in V_\alpha$ *relative to* f if it further satisfies that there is a neighborhood U of $x \in X$, a complex analytic embedding of $(U, x) \hookrightarrow \mathbb{C}^m$ and an analytic extension $\tilde{f}$ of f to a neighborhood of x in $\mathbb{C}^m$, such that for every sequence $(x_i)_{i \in I}$ of points in V_β that converge to x for which the sequence of tangent spaces $T_{x_i}(\tilde{f}^{-1}(\tilde{f}(x_i) \cap V_\beta)$ has a limit T, then this limit contains the tangent space at x of $\tilde{f}^{-1}(\tilde{f}(x) \cap V_\alpha)$, *i.e.*,

$$T_x\big(f^{-1}(\tilde{f}(x) \cap V_\alpha)\big) \subset T.$$

Definition 3.5.3. We say that the Whitney stratification $(V_\alpha)_{\alpha \in A}$ satisfies the *Thom a_f condition* if it satisfies the above Thom condition relative to f for each pair of strata (V_β, V_α) with $V_\alpha \subset \overline{V}_\beta$.

So the Thom a_f condition is in some sense like a Whitney (a) condition relative to the fibers of the morphism f. Now we need to introduce a finer condition. For details we refer to [81] where this concept is defined and studied in a very general setting. For simplicity we restrict our discussion to what we need for this work.

We recall the distance δ defined for two linear subspaces E and F in $\mathbb{C}^m$ by:

$$\delta(E, F) = \sup_{\substack{u \in E \setminus \{0\} \\ v \in F \setminus \{0\}}} \Big(\frac{|\langle u, v \rangle|}{\|u\| \|v\|} \Big).$$

Let U be an open neighborhood of the origin in $\mathbb{C}^m = \mathbb{C}^{n+k}$ and consider a holomorphic map

$$f : (U, 0) \longrightarrow (\mathbb{C}^k, 0) \ , \ m > k \geq 1,$$

defining a complete intersection germ $V := f^{-1}(0)$ of dimension n. Let $(V_\alpha)_{\alpha\in A}$ be a Whitney stratification of V, let $\varDelta$ be the discriminant of f and set $U_0 = U \setminus f^{-1}(\varDelta)$.

Definition 3.5.4. Given a stratum V_α we say that the pair (U_0, V_α) satisfies *the strict Thom w_f condition* if for every $x_o \in V_\alpha$ there is a neighborhood U_{x_o} in U and a constant C such that for every $x \in U_{x_o} \setminus V$ one has:

$$\delta(T_{x_o}V_\alpha, T_x f^{-1}(f(x))) \leq C \cdot d(x, x_o),$$

where d is the Euclidian distance in $\mathbb{C}^{n+k}$ and δ is the distance between linear subspaces of $\mathbb{C}^{n+k}$ defined above.

Definition 3.5.5. We say that the Whitney stratification $(V_\alpha)_{\alpha\in A}$ of X satisfies the *strict Thom w_f condition* if it satisfies the above w_f condition for all strata.

Notice that the a_f condition can be expressed saying that for each sequence $x_i \in U \setminus \{X\}$ converging to x_o one has, if the limit exists,

$$\lim_{i\to\infty} \delta\big(T_{x_o}V_\alpha, T_{x_i} f^{-1}(f(x_i))\big) = 0.$$

Hence it is clear that the w_f condition implies the a_f condition. They are actually equivalent in the complex analytic setting, by Teissier's work [163].

Consider again a holomorphic map

$$f : (U, 0) \longrightarrow (\mathbb{C}^k, 0) \ , \ m > k \geq 1,$$

defining a complete intersection germ $V := f^{-1}(0)$ of dimension n, and assume there is a Whitney stratification of V that satisfies the Thom a_f or w_f *condition* for all strata. Let $\varDelta$ be the discriminant of f and set $U_0 = U \setminus f^{-1}(\varDelta)$. Then:

Theorem 3.5.1. *For every $\varepsilon > 0$ sufficiently small and $\delta = \delta(\varepsilon) > 0$ sufficiently small with respect to ε, the map:*

$$f : \big(\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta)\big) \setminus f^{-1}(\varDelta) \longrightarrow (\mathbb{D}_\delta \setminus \varDelta) \subset \mathbb{C}^k,$$

where $\mathbb{B}_\varepsilon$ is a small ball around $0 \in \mathbb{C}^{n+k}$ and $\mathbb{D}_\delta$ is a small ball around $0 \in \mathbb{C}^k$, is a locally trivial topological fibration.

The crucial point is to notice that the Thom a_f condition guarantees that the fibers $f^{-1}(t)$ intersect transversely the boundary sphere $\mathbb{S}_\varepsilon$ and therefore one may apply the first Thom–Mather Isotopy Theorem to get a fibration. For that one can follow the indications given by Lê in his proof of the fibration theorem [102]. An alternative proof of this theorem follows from Verdier's work [167] about rugose vector fields and the Thom–Mather theorems.

3.5.2 The Hypersurface Case

Let us now denote by $(V, 0)$ a hypersurface in an open set $U \subset \mathbb{C}^{n+1}$ defined by a holomorphic function $f : (U, 0) \to (\mathbb{C}, 0)$. We endow V with a Whitney stratification $(V_\alpha)_{\alpha \in A}$. By [128] or [37] we can assume that the stratification satisfies the w_f condition.

Let us consider the subspace $\mathcal{E}$ of the tangent bundle TU of U consisting of the union of the tangent bundles of all the strata. We recall (Chap. 2) that a *stratified vector field* on V means a section of TU whose image is in $\mathcal{E}$.

Let v be a stratified vector field on $(V, 0)$ with an isolated singularity (zero) at $0 \in V$. We want to define an index of v at $0 \in V$ in the spirit of the *GSV*-index. We recall that if V has an isolated singularity at 0 this index is equal to the Poincaré–Hopf index of an extension of v to the Milnor fiber $\mathbf{F}$ of f; in particular this index is $\chi(\mathbf{F})$ if v is radial.

For this, let us consider a (sufficiently small) ball $\mathbb{B}_\varepsilon$ around $0 \in U$ and denote by $\mathcal{T}$ the Milnor tube $f^{-1}(\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$, where $\mathbb{D}_\delta$ is a (sufficiently small) disk around $0 \in \mathbb{C}$. We let $\partial\mathcal{T}$ be the "boundary" $f^{-1}(\mathbb{S}_\delta) \cap \mathbb{B}_\varepsilon$ of $\mathcal{T}$, $\mathbb{S}_\delta = \partial\mathbb{D}_\delta$.

Let r be the radial vector field in $\mathbb{C}$ whose solutions are straight lines converging to 0. By [167], it can be lifted to an integrable vector field $\widetilde{r}$ in $\mathcal{T}$, whose solutions are arcs that start in $\partial\mathcal{T}$, they finish in V and they are transverse to all the tubes $f^{-1}(\mathbb{S}_\eta)$ with $\eta \in]0, \delta[$. This vector field $\widetilde{r}$ defines a C^∞ retraction ξ of $\mathcal{T}$ into V, with V as fixed point set. The restriction of ξ to any fixed Milnor fiber $\mathbf{F} = f^{-1}(t_0) \cap \mathbb{B}_\varepsilon,\ t_0 \in \mathbb{S}_\delta$, provides a continuous map $\pi : \mathbf{F} \to V$, which is surjective and it is C^∞ over the regular part of V. We call such map ξ, or also π, a *tube map* for V. Since the singular set $\mathrm{Sing}(V)$ of V is a Zariski closed subset of V, we notice that we can choose the lifting $\widetilde{r}$ so that $\pi^{-1}(V_{\mathrm{reg}})$ is an open dense subset of $\mathbf{F}$, where V_{reg} is the regular part $V_{\mathrm{reg}} = V \setminus \mathrm{Sing}(V)$.

We want to use π to lift the stratified vector field v on V to a vector field on $\mathbf{F}$. Firstly, let us consider the case where V has an isolated singularity at 0. The map π is a diffeomorphism restricted to a neighborhood $N \subset \mathbf{F}$ of $\mathbf{F} \cap \partial\mathbb{B}_\varepsilon$. Then v can be lifted to a nonsingular vector field on N and extended to the interior of $\mathbf{F}$ with finitely many singularities, by elementary obstruction theory. The total Poincaré–Hopf index of this vector field on $\mathbf{F}$ is the GSV index of v on V.

We want to generalize this construction to the case where the singularity of V at 0 is not necessarily isolated. Let us consider $(V, 0)$ as above, a possibly nonisolated germ. We fix a Milnor fiber $\mathbf{F} = f^{-1}(t_o) \cap \mathbb{B}_\varepsilon$ for some $t_o \in \mathbb{S}_\delta$. Given a point $x \in \mathbf{F}$, we let γ_x be the solution of $\widetilde{r}$ that starts at x. The end-point of γ_x is the point $\pi(x) \in V$. We parameterize this arc γ_x by the interval $[0, 1]$, with $\gamma_x(0) = x$ and $\gamma_x(1) = \pi(x)$. We assume that this interval $[0, 1]$ is the straight arc in $\mathbb{C}$ going from t_o to 0, so that for each $t \in [0, 1[$, the point $\gamma_x(t)$ is in a unique Milnor fiber $\mathbf{F}_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon$. The family of tangent spaces to $\mathbf{F}_t$ at the points $\gamma_x(t)$ define a 1-parameter family of n-dimensional subspaces of $\mathbb{C}^{n+1}$, that converges to an n-plane $\Lambda_{\pi(x)} \subset T_{\pi(x)}(U)$ when $t \to 1$; one has an induced isomorphism $T_x\mathbf{F} \cong \Lambda_{\pi(x)}$.

Since the stratification satisfies Thom's a_f condition, $\Lambda_{\pi(x)}$ contains the space $T_{\pi(x)}V_\alpha$, tangent to the stratum that contains $\pi(x)$. Hence, the given vector $v(\pi(x))$ can be lifted to a vector $\widetilde{v}(x) \in T_x\mathbf{F}$. Thus we obtain a vector field $\widetilde{v}$, nonsingular over the inverse image of $V \setminus \{0\}$, which is open and dense in $\mathbf{F}$. The w_f condition guarantees that this vector field $\widetilde{v}$ is continuous and nonzero on a neighborhood of $\mathbf{F} \cap \partial\mathbb{B}_\varepsilon$ since v is assumed to have an isolated singularity at 0. Thus $\widetilde{v}$ has a well defined Poincaré–Hopf index in $\mathbf{F}$. Furthermore, by the w_f condition the angle between $v(\pi(x))$ and $\widetilde{v}(x)$ is small. More precisely, given any $\alpha > 0$ small, we can choose δ sufficiently small with respect to α so that the angle between $v(\pi(x))$ and $\widetilde{v}(x)$ is less than α. This implies that if we replace $\widetilde{v}$ by some other lifting of v, the induced vector fields on $\mathbf{F}$ are homotopic. Since f induces a locally trivial fibration over the punctured disk $\mathbb{D}_\delta \setminus 0$, then the homotopy class of $\widetilde{v}$ does not depend on the choice of the Milnor fiber. So we obtain:

Proposition 3.5.1. *The Poincaré–Hopf index of $\widetilde{v}$ in $\mathbf{F}$ depends only on $V \subset U$ and the vector field v. It is independent of the choices of the Milnor fiber $\mathbf{F}$ as well as the liftings involved in its definition.*

Definition 3.5.6. We call this integer the GSV index of v on V and we denote it by $\mathrm{Ind}_{\mathrm{GSV}}(v, 0)$.

3.5.3 The Complete Intersection Case

We now consider a holomorphic map

$$f : (U, 0) \longrightarrow (\mathbb{C}^k, 0)\ ,\ m > k \geq 1,$$

on an open neighborhood of the origin in $\mathbb{C}^m = \mathbb{C}^{n+k}$, defining a complete intersection germ $V := f^{-1}(0)$ of dimension n. Let $\Delta \subset \mathbb{C}^k$ be the discriminant of f and set $U_0 = U \setminus f^{-1}(\Delta)$.

The constructions are similar to those for hypersurfaces. The main difference is that if $k > 1$ then there exists complete intersection germs which do not admit any stratification satisfying the a_f condition, and we actually need the w_f condition. So we must add this assumption. Thus for the rest of this section we assume we are given a Whitney stratification $(V_\alpha)_{\alpha \in A}$ of U, adapted to V, satisfying the w_f condition.

As before, let v be a stratified vector field on $(V, 0)$ with an isolated singularity at $0 \in V$. We want to define its GSV index.

Let us consider a small ball $\mathbb{B}_\varepsilon$ around $0 \in U$. Let v_{rad} be an integrable radial vector field in a sufficiently small (with respect to ε) disk $\mathbb{D}_\delta$ around $0 \in \mathbb{C}^k$, whose solutions are arcs converging to 0 and for which Δ is an invariant set. This is possible by [121, 141] because Δ is semi-analytic in $\mathbb{C}^k$. We can assume further that for each $t \in \mathbb{D}_\delta \setminus \Delta$ the (Milnor) fiber $\mathbf{F}_t = f^{-1}(t)$

intersects the boundary sphere $\partial\mathbb{B}_\varepsilon$ transversely. Set $\mathcal{T} = f^{-1}(\mathbb{D}_\delta \setminus \Delta)$. The w_f condition implies that the map

$$f|_{\mathcal{T}} \colon \mathcal{T} \to \mathbb{D}_\delta \setminus \Delta$$

is a locally trivial fibration and by [167] we can lift v_{rad} to vector field $\widetilde{r}$ in $\mathcal{T}$, whose solutions are arcs that start in $\partial\mathcal{T} = f^{-1}(\mathbb{S}_\delta \setminus \Delta)$, $\mathbb{S}_\delta = \partial\mathbb{D}_\delta$, they finish in V and they are transverse to all the "tubes" $f^{-1}(\mathbb{S}_\eta)$ with $\eta \in]0, \delta[$.

This vector field $\widetilde{v}_{\text{rad}}$ defines a C^∞ retraction ξ of $\mathcal{T}$ into V, with V as fixed point set. The restriction of ξ to any fixed Milnor fiber $\mathbf{F} = f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$, $t_0 \in \mathbb{S}_\delta$, provides a continuous map $\pi : \mathbf{F} \to V$ which is surjective and it is C^∞ over the regular part of V. As before, we call such map ξ, or also π, a *tube map* for V.

We use π to lift the stratified vector field v on V to a vector field on the fixed Milnor fiber $\mathbf{F}$. Given a point $x \in \mathbf{F}$, we let γ_x be the solution of $\widetilde{v}_{\text{rad}}$ that starts at x. The end-point of γ_x is the point $\pi(x) \in V$. We parameterize this arc γ_x by the interval $[0, 1]$, with $\gamma_x(0) = x$ and $\gamma_x(1) = \pi(x)$. We assume that this interval $[0, 1]$ is the arc in $\mathbb{D}_\delta$ going from t_o to 0, so that for each $t \in [0, 1[$ the point $\gamma_x(t)$ is in a unique Milnor fiber $\mathbf{F}_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon$. The family of tangent spaces to $\mathbf{F}_t$ at the points $\gamma_x(t)$ define a 1-parameter family of n-dimensional subspaces of $\mathbb{C}^{n+k}$ that converges to an n-plane $\Lambda_{\pi(x)} \subset T_{\pi(x)}(U)$ when t tends to 1; one has an induced isomorphism $T_x\mathbf{F} \simeq \Lambda_{\pi(x)}$.

Just as for hypersurfaces, since the stratification satisfies Thom's a_f condition, $\Lambda_{\pi(x)}$ contains the space $T_{\pi(x)}V_\alpha$ tangent to the stratum that contains $\pi(x)$. Hence, the given vector $v(\pi(x))$ can be lifted to a vector $\widetilde{v}(x) \in T_x\mathbf{F}$. Thus we obtain a vector field $\widetilde{v}$, nonsingular over the inverse image of $V \setminus \{0\}$, which is open and dense in $\mathbf{F}$, and this vector field is continuous by the w_f condition; it is also nonzero on a neighborhood of $\mathbf{F} \cap \partial\mathbb{B}_\varepsilon$ since v is assumed to have an isolated singularity at 0. Thus $\widetilde{v}$ has a well defined Poincaré–Hopf index in $\mathbf{F}$. As before, the homotopy class of $\widetilde{v}$ does not depend on the several choices involved and we have:

Proposition 3.5.2. *The Poincaré–Hopf index of $\widetilde{v}$ in $\mathbf{F}$ depends only on $V \subset U$ and the vector field v. It is independent of the choices of the Milnor fiber $\mathbf{F}$ as well as the liftings involved in its definition.*

Definition 3.5.7. We call this integer the GSV index of v on V and we denote it by $\mathrm{Ind}_{\mathrm{GSV}}(v, 0)$.

3.6 The Proportionality Theorem

We consider again a holomorphic map

$$f : (U, 0) \longrightarrow (\mathbb{C}^k, 0)\ ,\ m > k \geq 1,$$

defining a complete intersection germ $V := f^{-1}(0)$ of dimension n and we assume $(V_\alpha)_{\alpha\in A}$ is a Whitney stratification of U adapted to V satisfying the w_f condition. As we know, such a stratification always exists if $k = 1$; for $k > 1$ we must assume its existence. Given a stratified vector field v on V with an isolated singularity at 0 we have the index $\mathrm{Ind}_{\mathrm{GSV}}(v, 0)$ defined above. This is by definition the Poincaré–Hopf index of a lifting of v to a Milnor fiber of f.

The goal now is to relate this index with other invariants of v and V. We cannot yet answer this in general, but the theorem below gives an answer for vector fields which are obtained by radial extension. We remark that in [65] there is defined an index for holomorphic vector fields on varieties with nonisolated singularities; it is likely that in the hypersurface case the two indices coincide (see Chap. 7 below for more on this subject).

Let us consider first the case where $v = v_{\mathrm{rad}}$ is a stratified *radial* vector field, *i.e.*, it is transverse to the boundary $\partial\mathbb{B}_\varepsilon$ of every small ball $\mathbb{B}_\varepsilon$, *pointing outwards*; it has a unique singular point (inside $\mathbb{B}_\varepsilon$) at 0. The Poincaré–Hopf index of w at the point 0, denoted by $\mathrm{Ind}_{\mathrm{PH}}(v_{\mathrm{rad}}, 0)$, is equal to 1, computed either in the stratum V_α of V containing 0 (if $\dim V_\alpha > 0$) or in the ambient space $\mathbb{C}^{n+k}$. The lifting $\widetilde{v}_{\mathrm{rad}}$ is a vector field on $\mathbf{F}$ transverse to the boundary $\partial\mathbf{F}$, since the angle between $\widetilde{v}_{\mathrm{rad}}(x)$ and $v_{\mathrm{rad}}(x)$ is small (by the w_f condition). Thus we obtain:

Proposition 3.6.1. *If v_{rad} is a stratified radial vector field pointing outwards the ball $\mathbb{B}_\varepsilon$ along its boundary $\partial\mathbb{B}_\varepsilon$, then its GSV index equals the Euler–Poincaré characteristic of the Milnor fiber* $\mathbf{F}$*:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0) = \chi(\mathbf{F})$$

Let us consider now a stratified vector field v in general, defined on the ball $\mathbb{B}_\varepsilon \subset U$, with a unique singularity at 0.

It follows from Proposition 1.1.1 that if v is a vector field obtained by radial extension, then the Poincaré–Hopf index of v computed in the stratum V_α equals the Poincaré–Hopf index of v computed in $\mathbb{C}^{n+k}$ (and this number is the Schwartz index by definition). If the stratum containing x_0 has dimension 0, then this index is $+1$.

We notice that we can always perturb the restriction of v on V_α to obtain a new vector field w on V_α. Under such a perturbation x_o splits into a number $x_1, \ldots, x_q$ of singularities of w. Then the Poincaré–Hopf index of $v|_{V_\alpha}$ at x_o equals the sum of the Poincaré–Hopf indices of w at $x_1, \ldots, x_q$, by the stability of the Poincaré–Hopf index. We can extend w to a neighborhood of x_o in V by radial extension, using M.-H. Schwartz technique, and obtain a perturbation of v in a neighborhood of x_o in V. In this case we have that the Schwartz index of v at x_o is the sum of the Schwartz indices at $x_1, \ldots, x_q$. Similarly one has the following lemma:

Lemma 3.6.1. *(Stability of the index). Suppose v is obtained by radial extension in a neighborhood of $x_o \in V_\alpha$. Let w be a stratified vector field on V*

obtained by a small perturbation of v *in the stratum* V_α *and extending this to a neighborhood of* x_o *by radial extension. Let* $x_1, \ldots, x_q$ *be the singularities of* w *into which* x_o *splits under this perturbation. Then:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v, x_o) = \sum_{i=1}^{q} \mathrm{Ind}_{\mathrm{GSV}}(w, x_i)$$

Proof. Consider the tube map $\pi : \mathbf{F} \to V$ where $\mathbf{F}$ is a local Milnor fiber of V at x_o. We know that this map lifts v to a vector field $\tilde{v}$ on $\mathbf{F}$, non singular near the boundary $\partial\mathbf{F}$. By definition $\mathrm{Ind}_{\mathrm{GSV}}(v, x_o)$ is the Poincaré–Hopf index of $\tilde{v}$ in $\mathbf{F}$, and we know that this number is independent of the way we extend $\tilde{v}$ to the interior of $\mathbf{F}$, by Theorem 1.1.2. Thus the idea is to choose this extension appropriately: we start by perturbing v as in the statement of Lemma 3.6.1 and lifting w to a vector field $\tilde{w}$ on $\mathbf{F}$ which coincides with $\tilde{v}$ near $\partial\mathbf{F}$. By Theorem 1.1.2 the total Poincaré–Hopf index of $\tilde{w}$ in $\mathbf{F}$, $\mathrm{Ind}_{\mathrm{PH}}(\tilde{w}, \mathbf{F})$, equals $\mathrm{Ind}_{\mathrm{GSV}}(v, x_o)$. But

$$\mathrm{Ind}_{\mathrm{PH}}(\tilde{w}, \mathbf{F}) = \sum_{i=1}^{q} \mathrm{Ind}_{\mathrm{GSV}}(w, x_i)$$

by construction. Hence the lemma.

As corollary we obtain the Proportionality Theorem for vector fields of [34]. An alternative proof is given in [28] in the spirit of that in [33]. The present proof is reminiscent of M.-H. Schwartz' proof of Théorème 4.2.3 in [141].

Theorem 3.6.1. *Let* v *be a stratified vector field in* V *obtained by radial extension in a neighborhood of the singularity* $x_o \in V_\alpha \subset V$. *Then the GSV index of* v *at* $x_o \in V$, $\mathrm{Ind}_{\mathrm{GSV}}(v, x_o)$, *is proportional to the local Poincaré–Hopf index* $\mathrm{Ind}_{\mathrm{PH}}(v, x_o)$ *of* v *at* x_o *(regarded as a vector field in* V_α*):*

$$\mathrm{Ind}_{\mathrm{GSV}}(v, x_o) = \mathrm{Ind}_{\mathrm{PH}}(v, x_o) \cdot \chi(\mathbf{F})$$

where $\mathbf{F}$ *is the Milnor fiber of* $\mathbf{F}$.

Proof. If $\mathrm{Ind}_{\mathrm{PH}}(v, x_o) = 1$ then v is homotopic to a radial vector field and the claim follows from Proposition 3.6.1. Suppose now that $\mathrm{Ind}_{\mathrm{PH}}(v, x_o) = -1$. Let D_{x_o} be a small disk in V_α around x_o. By [153] we can always extend $v|_{V_\alpha}$ to a vector field w on a bigger disk $\widehat{D}$ in V_α containing D_{x_o}, so that w is transverse to the boundary of $\widehat{D}$, pointing outwards, and it has exactly three singular points in $\widehat{D}$: x_o, where the local index is -1 by hypothesis, and two other points x_1, x_2 of local index 1. We may now construct a 1-parameter family of vector fields on an open disk in V_α which collapses these three singularities into a single one of index 1 at x_o; we denote the resulting vector field by $\widehat{v}$. And we now extend all these vector fields by radial extension. By Proposition 3.6.1 and Lemma 3.6.1 one has:

$$\chi(\mathbf{F}) = \mathrm{Ind}_{\mathrm{GSV}}(\widehat{v}, x_o) = \sum_{i=0}^{2} \mathrm{Ind}_{\mathrm{GSV}}(w, x_i) = \mathrm{Ind}_{\mathrm{GSV}}(v, x_o) + 2\chi(\mathbf{F}).$$

Hence $\mathrm{Ind}_{\mathrm{GSV}}(v, x_o) = -\chi(\mathbf{F})$ and the theorem is proved when $\mathrm{Ind}_{\mathrm{PH}}(v, x_o) = -1$. The general case follows easily: given v, we can always "morsify" its restriction to V_α and extend the morsification by radial extension, to get a vector field w whose singularities have all local Poincaré–Hopf indices ± 1 in V_α. Thus the theorem follows from Proposition 3.6.1, Lemma 3.6.1 and the above proof for the case of local index -1.

3.7 Geometric Applications

In this section we give applications of the GSV index to three different problems in geometry. The first gives a proof of a theorem by B. Teissier about invariance of the Milnor number for algebraic knots; the second discusses the triviality of the bundle that defines the canonical contact structure on complex hypersurface germs; the third discusses the triviality of the normal bundle on the regular part of a holomorphic foliation in the neighborhood of an isolated singularity.

3.7.1 Topological Invariance of the Milnor Number

It was shown by Teissier in [162] that the Milnor number of hypersurface singularities is determined by the corresponding algebraic knot $(\mathbb{S}_\varepsilon, \mathbf{K})$. That is,

Theorem 3.7.1. *If two hypersurface germs $(V_1, 0)$, $(V_2, 0)$ in $\mathbb{C}^{n+1}$ are such that for sufficiently small spheres $\mathbb{S}_{\varepsilon_1}, \mathbb{S}_{\varepsilon_2}$ the pairs $(\mathbb{S}_{\varepsilon_1}, \mathbf{K}_1)$ and $(\mathbb{S}_{\varepsilon_2}, \mathbf{K}_2)$ are orientation preserving homeomorphic, then $\mu(V_1) = \mu(V_2)$.*

For $n > 2$ this can be proved using the GSV index. For this we will use the following Proposition:

Proposition 3.7.1. *Let $(\mathbb{S}_\varepsilon, \mathbf{K})$ be an algebraic knot defined by a hypersurface germ $(V, 0)$. Let v_{rad} be the restriction to $\mathbf{K}$ of the unit outwards normal vector field of $\mathbb{S}_\varepsilon$ in $\mathbb{C}^{n+1}$; let τ be some (any) nowhere-zero section of the normal bundle $\nu(\mathbf{K})$ of $\mathbf{K}$ in $\mathbb{S}_\varepsilon$ (which is a trivial bundle). Then the degree of the map*

$$(v_{\mathrm{rad}}, \tau) : \mathbf{K} \longrightarrow W_{2,n+1},$$

into the Stiefel manifold is an invariant of $\mathbf{K}$, equal to

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0) = 1 + (-1)^n \mu(V).$$

Proof. It is clear that v_{rad} is homotopic to a radial vector field tangent to V. On the other hand, notice that the bundle $\nu(\mathbf{K})$ has complex dimension 1. Hence every two never-zero sections of this bundle differ, up to homotopy, by a map $\mathbf{K} \to U(1) \simeq \mathbb{S}^1$. If $n > 2$ the link $\mathbf{K}$ is simply connected and therefore every such map is nulhomotopic. This implies that every never-zero section of $\nu(\mathbf{K})$ is homotopic (through never-zero sections) to the (complex conjugate) gradient vector field of some function that defines the germ of V. The result now follows from Theorem 3.2.1.

Now, given an orientation preserving diffeomorphism $h : (\mathbb{S}_{\varepsilon_1}, \mathbf{K}_1) \to (\mathbb{S}_{\varepsilon_2}, \mathbf{K}_2)$ between algebraic knots, this carries the vector field v_{rad} of the first pair into a vector field which is necessarily transverse to the second sphere, and therefore homotopic to the radial vector field. This also carries the normal section of $\mathbf{K}_1$ in the sphere into a normal section of $\mathbf{K}_2$; therefore the two singularities have same Milnor number by the proposition above.

If the map h above is only a homeomorphism and not a diffeomorphism, one can argue as in [71] to show that one can replace the vector fields in the statement above by the corresponding flows, and a homeomorphism carries flows into flows. The idea is very simple: the given vector field defines a local flow γ_t with no stationary points. Then choose a fixed time $t_o > 0$ so that the flow is defined at each point of $\mathbf{K}_1$ at least for time $t_o > 0$ (this exists by compactness). Now, for each $x \in \mathbf{K}_1$ take the oriented secant that joins x and $\gamma_{t_o}(x)$; this defines a vector field on $\mathbb{C}^{n+1}$ restricted to $\mathbf{K}_1$, which is of course homotopic to the original vector field. Now consider the flow $h\gamma_t h^{-1}$. The previous construction produces a vector field on $\mathbf{K}_2$. If we started with a radial vector field on $\mathbf{K}_1$, the new vector field will be homotopic to the radial vector field on $\mathbf{K}_2$. If we started with a normal vector field on $\mathbf{K}_1$, the new vector field can be turned by a homotopy into a normal vector field for $\mathbf{K}_2$. This proves the statement, *i.e.*, that the Milnor number depends only on the corresponding algebraic knot.

3.7.2 The Canonical Contact Structure on the Link

In this subsection, we intend to provide an application to the contact geometry, that is based on [146]. The main result is the following Theorem 3.7.2.

Let $V \subset \mathbb{C}^m$ be a complex analytic variety of dimension $n > 1$ with an isolated singularity at $0 \in \mathbb{C}^m$. It is well-known that the diffeomorphism type of its link $\mathbf{K} = V \cap \mathbb{S}_e$ does not depend on the choices of the embedding of V in $\mathbb{C}^m$ nor on the sphere $\mathbb{S}_\varepsilon$, provided this is small enough. Moreover, according to [166] one has a natural contact structure $\mathcal{C}_V$ on $\mathbf{K}$, which is again independent of the embedding of V in $\mathbb{C}^m$ and the choice of the sphere, up to contactmorphism. We refer to $\mathcal{C}_V$ as the *canonical contact structure* on $\mathbf{K}$.

To define this contact structure, notice that the normal bundle $\nu(\mathbf{K})$ of $\mathbf{K}$ in V has a canonical trivialization given by the unit, outwards-pointing vector field τ of $\mathbf{K}$ in V, which is the restriction to $\mathbf{K}$ of a radial vector field $v_{\rm rad}$ on V. The complex orthogonal complement $v_{\rm rad}^{\perp}$ of $v_{\rm rad}$ at each point in $V^* = V \setminus \{0\}$ is an $(n-1)$-dimensional complex plane in the tangent bundle of TV^*. A theorem of Varchenko establishes that the restriction of $v_{\rm rad}^{\perp}$ to the link $\mathbf{K}$, that we denote $\mathcal{D}_V$, determines the canonical contact structure $\mathcal{C}_V$ on $\mathbf{K}$. If we set $\sqrt{-1} = i$ as usual, then the vector field $i \cdot \tau$ is, up to scaling, the Reeb vector field of the contact structure.

For example, if $n = 2$ and the germ $(V, 0)$ is an ICIS, then one has a nowhere-vanishing holomorphic 2-form Ω around 0 in V; if we equip V^* with the hermitian metric induced from that in $\mathbb{C}^m$, then the 2-form Ω determines a reduction of the structure group of $T(V^*)$ from $U(2)$ to $SU(2) \cong Sp(1)$, so it determines an $Sp(1)$-structure on the complex bundle $T(V^*)$ (see [143]). If, as before, we denote by τ the unit outwards normal vector field of $\mathbf{K}$ in V, then the bundle $\mathcal{D}_V$ is the trivial 1-dimensional complex bundle spanned by the vector field $j \cdot \tau$, obtained by multiplying the vector $\tau(x)$ by the quaternion j at each point of $\mathbf{K}$.

Here we give a necessary and sufficient condition for $\mathcal{D}_V$ to be a trivial bundle when $n > 2$ and the germ of V at 0 is an ICIS:

Theorem 3.7.2. *The complex bundle $\mathcal{D}_V$ that defines the canonical contact structure on $\mathbf{K}$ is C^∞ trivial as a complex vector bundle if and only if the Milnor number $\mu(V, 0)$ of the ICIS germ $(V, 0)$ satisfies:*

$$\mu(V, 0) \equiv (-1)^{n-1} \qquad mod\,(n-1)!,$$

equivalently, the Euler–Poincaré characteristic of the Milnor fiber satisfies

$$\chi(\mathbf{F}) \equiv 0 \qquad mod\,(n-1)!.$$

For example, in the case of the quadric $V = \{z_1^2 + \cdots + z_{n+1}^2 = 0\}$ in $\mathbb{C}^{n+1}$, the bundle $\mathcal{D}_V$ is trivial if and only if $n = 2$ or n is an odd number.

Recall that V has an associated Milnor fibration [121], and Milnor proved that the Milnor fiber $\mathbf{F}$ can be regarded as a compact manifold with boundary the link $\mathbf{K}$, and $\mathbf{F}$ is a parallelizable manifold with the homotopy type of a bouquet of spheres of middle dimension, the number of spheres in this bouquet being the Milnor number. So its Euler–Poincaré characteristic is $\chi(\mathbf{F}) = 1 + (-1)^n \mu(V, 0)$. Then Theorem 3.7.2 essentially follows from the following Theorem 3.7.3 applied to a vector field which is everywhere transversal to the link $\mathbf{K}$. We recall that given a vector field v on V, singular only at 0, its GSV index equals the Poincaré–Hopf index of an extension of v to a Milnor fiber. Thus, Theorem 3.7.2 can be rephrased by saying that the complex orthogonal complement of τ in $T(V \setminus 0)$ is a trivial bundle if and only if the GSV index of v is a multiple of $(n-1)!$ (see [146] for details). This will follow from the Theorem below taking as the manifold W the Milnor fiber $\mathbf{F}$.

Theorem 3.7.3. *Let W be a $2n$-dimensional, $n \geq 1$, compact, connected manifold with nonempty boundary ∂W and trivial tangent bundle; moreover, fix a trivialization $v_0^{(n)} : TW \to W \times \mathbb{C}^n$ and equip TW with the hermitian metric induced from that in $\mathbb{C}^n$. Assume further that W has the homotopy type of a bouquet of n-spheres. Let v be a nowhere-zero, continuous vector field on a neighborhood of ∂W in W. Then the complex orthogonal complement of v in $TW|_{\partial W}$ is a C^∞ trivial complex bundle if and only if v extends to the interior of W with total Poincaré–Hopf index a multiple of $(n-1)!$.*

We work always in the category of topological spaces and continuous maps, so the proofs of these theorems actually discuss topological triviality of the vector bundles in question. But everything becomes automatically C^∞ because every continuous map between smooth manifolds can be approximated by a smooth map.

Theorem 3.7.3 is a consequence of the following two lemmas 3.7.1 and 3.7.2:

Lemma 3.7.1. *Let W be a $2n$-dimensional, $n \geq 1$, compact, connected manifold with nonempty boundary ∂W and trivial tangent bundle, trivialized by a complex n-frame $v_0^{(n)} : TW \to W \times \mathbb{C}^n$. Let v be a continuous vector field, defined and nonsingular on a neighborhood of ∂W in W. If $\mathrm{Ind}_{\mathrm{PH}}(v, W)$ is a multiple of $(n-1)!$, then v can be completed to a continuous trivialization of the complex vector bundle $TW|_{\partial W}$. That is, there exist $(n-1)$ continuous sections $v_2, ..., v_n$ of $TW|_{\partial W}$, such that the set $\{v, v_2, ..., v_n\}$ defines a trivialization of $TW|_{\partial W}$.*

Lemma 3.7.2. *Let W be as above and assume further that W has the homotopy type of a bouquet of n-spheres. Let v be a continuous section of $TW|_{\partial W}$ which can be completed to a trivialization of the complex bundle $TW|_{\partial W}$; i.e., there exist continuous sections $v_2, ..., v_n$ of $TW|_{\partial W}$ such that the n-frame $v^{(n)} = \{v, v_2, ..., v_n\}$ defines a trivialization of $TW|_{\partial W}$ as a complex vector bundle. Then $\mathrm{Ind}_{\mathrm{PH}}(v, W)$ is a multiple of $(n-1)!$.*

The proofs of these lemmas are a little technical and they are given in detail in [146]. Here we explain only the main ideas. To motivate these, we restrict first to the case where W is the usual $2n$-ball $\mathbb{B}^{2n}$ with boundary $\mathbb{S}^{2n-1}$. This explains where the term $(n-1)!$ comes from.

We recall there is a classical fibration

$$U(n-1) \hookrightarrow U(n) \longrightarrow \mathbb{S}^{2n-1},$$

and an associated long exact homotopy sequence,

$$\cdots \to \pi_{2n-1}(U(n)) \xrightarrow{p_*} \pi_{2n-1}(\mathbb{S}^{2n-1}) \to \pi_{2n-2}(U(n-1)) \to \pi_{2n-2}(U(n)) \to \cdots \quad (3.7.4)$$

We know that $\pi_{2n-1}(\mathbb{S}^{2n-1}) \cong \mathbb{Z}$, and Bott's calculations in [18] tell us that:

(1) $\pi_{2n-1}(U(n)) \cong \mathbb{Z}$,

(2) $\pi_{2n-2}(U(n-1)) \cong \mathbb{Z}/(n-1)!$,

(3) $\pi_{2n-2}(U(n)) \cong 0$ and p_* is multiplication by $(n-1)!$.

Now observe that every continuous vector field on $\mathbb{B}^{2n}$ which is nonsingular away from the origin defines an element in $\pi_{2n-1}(\mathbb{S}^{2n-1})$. And conversely, every element in this homotopy group determines a homotopy class of vector fields on $\mathbb{B}^{2n}$, which are nonsingular away from the origin, and these are classified (up to homotopy) by their local Poincaré–Hopf index at 0. In other words, we can think of $\pi_{2n-1}(\mathbb{S}^{2n-1})$ as being (up to homotopy) the set of vector fields on W which are nonsingular on $\mathbb{S}^{2n-1}$, and these are classified by their local Poincaré–Hopf index at 0. Of course, homotopy of vector fields means homotopy through never vanishing vector fields.

Let us now look at the group $\pi_{2n-1}(U(n))$ and recall the classical construction of "twisting a framing" by Kervaire in [90] (see also [91]). Equip the tangent bundle $T\mathbb{B}^{2n}|_{\mathbb{S}^{2n-1}}$ with a complex trivialization $v_0^{(n)} : T\mathbb{B}^{2n}|_{\mathbb{S}^{2n-1}} \to \mathbb{S}^{2n-1} \times \mathbb{C}^n$, which we assume to be given by the usual basis of $T\mathbb{C}^n$. We further assume, for simplicity, that all frames here are unitary. Let $[\gamma]$ be an element in $\pi_{2n-1}(U(n))$ and $\gamma : \mathbb{S}^{2n-1} \to U(n)$ a representative of $[\gamma]$. Then, for each $x \in \mathbb{S}^{2n-1}$, $\gamma(x)$ is a linear transformation of $\mathbb{C}^n$, which carries the basis determined by $v_0^{(n)}(x)$ into a new basis that we may denote by $\gamma_*(v_0^{(n)})(x)$. Doing this for all points in $\mathbb{S}^{2n-1}$ we get a new n-frame $\gamma_*(v_0^{(n)})$ on $\mathbb{S}^{2n-1}$.

Conversely, given the frame $v_0^{(n)}$ as above, and another unitary n-frame $v^{(n)}$ on $\mathbb{S}^{2n-1}$, these two framings differ at each point $x \in \mathbb{S}^{2n-1}$ by an element in $U(n)$. Hence $v^{(n)}$ can be obtained as above, by twisting the frame $v_0^{(n)}$ by an appropriate map $\mathbb{S}^{2n-1} \to U(n)$. Therefore one has the following well-known theorem (see Kervaire's article for details):

Theorem. The homotopy classes of unitary frames on $\mathbb{S}^{2n-1}$ form a group, isomorphic to $\pi_{2n-1}(U(n))$.

We now observe that with these interpretations of $\pi_{2n-1}(\mathbb{S}^{2n-1})$ and $\pi_{2n-1}(U(n))$, both isomorphic to $\mathbb{Z}$, one has that the map p_* in (3.7.4) can be regarded as the map that associates to each unitary frame on $\mathbb{S}^{2n-1}$ the Poincaré–Hopf index in the ball $\mathbb{B}^{2n}$ of one of the n sections that define this frame (all such sections have the same local index because they are linearly independent everywhere).

Lemmas 3.7.1 and 3.7.2 then follow, for the case $W = \mathbb{B}^{2n}$, from the exact sequence (3.7.4) and Bott's computations in [18], implying that p_* is multiplication by $(n-1)!$.

Now in general, for W as in 3.7.1, since W is parallelizable and has nonempty boundary, there is an immersion $\imath : W \to \mathbb{R}^{2n} \cong \mathbb{C}^n$, by the immersion theorem of Hirsch–Poenarú (see [132]). Thus one has an induced (Gauss-type) continuous map $\partial W \xrightarrow{\psi_v} \mathbb{S}^{2n-1}$, defined by

$$\psi_v(x) \;=\; \frac{D\imath(v(x))}{|D\imath(v(x))|},$$

where D is the derivative.

By obstruction theory (see [153]), v can be extended to all of W minus one point, say x_o, around which $\imath$ is an embedding. Thus ψ_v extends to a map $W \setminus \{x_o\} \to \mathbb{S}^{2n-1}$. Hence the topological degree of ψ_v equals $\mathrm{Ind}(v, W)$. Moreover, by Hopf's theorem, two maps $\partial W \to \mathbb{S}^{2n-1}$ are homotopic if and only if they have the same degree. Thus one has that some other vector field v' on a neighborhood of ∂W in W is homotopic to v (through never-vanishing vector fields) if and only if $\mathrm{Ind}(v', W) = \mathrm{Ind}(v, W)$.

Now we assume that $\mathrm{Ind}(v, W)$ is a multiple of $(n-1)!$, *i.e.*, $\mathrm{Ind}(v, W) = t(n-1)!$ for some integer t. Let w be a vector field on W with index t and nonvanishing on ∂W (since W has nonempty boundary ∂W, one has on W vector fields with all possible Poincaré–Hopf total indices and never-zero on ∂W). Following [90, 91], twist the trivialization $v_0^{(n)}$ on the boundary ∂W as before, using the corresponding map ψ_w obtained via an immersion of W in $\mathbb{C}^n$; we get a new trivialization $v^{(n)} = (\psi_w)_*(v_0^{(n)})$ of $TW|_{\partial W}$. This means that at each point $x \in \partial W$ we change the basis of T_xW given by $v_0^{(n)}$ into its image by the linear map $\psi_w(x) \in U(n)$. We claim that $v_0^{(n)}$ has v as one of its n sections, up to homotopy; this will complete the proof of the lemma.

To prove the above claim notice first that, by the previous discussion, ψ_w has degree t. This implies that the trivialization $v_0^{(n)}$ of $TW|_{\partial W}$ represents the element $t \cdot [\gamma]$ of $\pi_{2n-1}(U(n)) \cong \mathbb{Z}$, where $[\gamma]$ is the positive generator of $\pi_{2n-1}(U(n))$. Then the exact sequence 3.7.4 implies that the map p_* carries the class represented by $v_0^{(n)}$ in $\pi_{2n-1}(U(n))$ into the class $t \cdot (n-1)! \cdot [\sigma]$ in $\pi_{2n-1}(\mathbb{S}^{2n-1})$, where $[\sigma]$ is the positive generator of this group, and we arrive to Lemma 3.7.1.

Now, for Lemma 3.7.2, we assume further that the manifold W^{2n} has the homotopy type of a bouquet of n-spheres, $n > 1$. The proof of Lemma 3.7.2 relies on a careful use of the relative Chern classes that we introduced in Chap. 1. We equip W with a triangulation compatible with the boundary ∂W, and we refer to $v^{(n)}$ as *a complex framing* on ∂W, meaning by this a trivialization of the complex bundle $TW|_{\partial W}$. We try to extend $v^{(n)}$ to the interior of W using the usual "stepwise" process: first to the 0-skeleton, then the 1-skeleton and so on, as far as we can.

According to Steenrod [153] (compare with Chap. 1), the successive obstructions to extending $v^{(n)}$ as a complex framing over the interior of W are elements in the relative cohomology $H^*(W, \partial W; \mathbb{Z})$. In fact these obstructions are cocycles that represent the Chern classes of W relative to the framing $v^{(n)}$ on ∂W. Thus they live in the even-dimensional relative cohomology of $(W, \partial W)$.

By Lefschetz duality one has $H^i(W, \partial W) \cong H_{2n-i}(W)$, hence all these groups vanish, except for $i = n, 2n$, since W is assumed to have the homotopy of a bouquet of n-spheres.

We now split the proof of 3.7.2 in two cases, according to the parity of n. Assume first n is odd. Since Chern classes live in even dimensions, in this case the only possible obstruction to extending $v^{(n)}$ to the interior of W is the top relative Chern class $c^n(W, v^{(n)}) \in H^{2n}(W, \partial W; \mathbb{Z})$. By definition, this class is the obstruction to extending to the interior of W one of the sections that define $v^{(n)}$, that we can take to be v. Hence $v^{(n)}$ can be extended to all of W minus one point, say x_o, and $\mathrm{Ind}_{\mathrm{PH}}(v, W)$ can be regarded as being both, the local Poincaré–Hopf index at x_o of the extension of v to $W \setminus \{x_o\}$, and also the Lefschetz dual $c^n(W, v^{(n)})[W, \partial W] \in H_0(W)$ of the Chern class $c^n(W, v^{(n)})$, where $[W, \partial W]$ is the fundamental cycle of the pair.

Since $v^{(n)}$ is already extended to a trivialization of $T(W \setminus \{x_o\})$, one has that $c^n(W, v^{(n)})$ can be identified with the Chern class of a small disk D_ε in W centered at x_o, relative to the framing $v^{(n)}$ on ∂D_ε. Then Lemma 3.7.2 follows in this case from the previous discussion for the case where W was a $2n$-disc.

Consider now the case n is even, say $n = 2m$, so W has real dimension $4m$. In this case one can prove the following lemma (see [146] for details):

Lemma 3.7.3. *The framing $v^{(n)}$ on ∂W extends to a trivialization $\widehat{v}^{(n)}$ of the complex bundle $T(W \setminus S_{v^{(n)}})$, where $S_{v^{(n)}}$ is an n-sphere embedded in the interior $\mathring{W}$ of W with trivial normal bundle. Hence $v^{(n)}$ extends to a trivialization of the complex bundle $T(W)$ away from the interior of a compact tubular neighborhood $\widehat{\mathbb{T}} \cong \mathbb{S}^n \times \mathbb{B}^n$ of $S_{v^{(n)}}$.*

By this lemma, $\widehat{v}^{(n)}$ is a complex framing that extends $v^{(n)}$ to all of W minus the interior $\mathrm{Int}\,\widehat{\mathbb{T}}$ of the solid torus $\widehat{\mathbb{T}} \cong \mathbb{S}^n \times \mathbb{B}^n$. We know from Chap. 1 that

$$c^n(W; v^{(n)})[W, \partial W] = \mathrm{Ind}_{\mathrm{PH}}(v, W),$$

where $[W, \partial W]$ is the fundamental cycle of the pair $(W, \partial W)$, and

$$c^n(W; v^{(n)})[W, \partial W] = c^n(\widehat{\mathbb{T}}, \widehat{v}^{(n)})[\widehat{\mathbb{T}}, \mathbb{T}]$$

because $\widehat{v}^{(n)}$ extends $v^{(n)}$. We claim that the latter integer is a multiple of $(n-1)!$, which obviously completes the proof of lemma 3.7.2. For this, recall $\pi_n(U(n)) = 0$ if $n > 1$ (see [18]), so we can assume that $v_o^{(n)}$ and $\widehat{v}^{(n)}$ coincide, up to homotopy, over a parallel $(\mathbb{S}^n \times *)$ of $\mathbb{T}$, where $*$ is a point in $\partial \mathbb{B}^n$. Using this one may now show that the complex framing $\widehat{v}^{(n)}$ is obtained from the trivialization $v_o^{(n)}$ of TW, twisting it in a neighborhood of a point, using Kervaire's construction (see [146] for details). So the lemma follows from the previous discussion for the case where W is a disc.

Remark 3.7.1. Notice that if a vector field v on the ICIS $(V, 0)$ is a component of a trivialization $v^{(n)}$ of the complex bundle TV^*, then the GSV index of

v equals (up to Lefschetz duality) the top Chern class of $\mathbf{F}$ relative to $v^{(n)}$. The theorem above shows that for $n > 2$ not all vector fields satisfy this condition, and therefore its GSV index cannot be always expressed as a Chern class of $T\mathbf{F}$ relative to a trivialization over $\partial\mathbf{F}$. However, if $f_1, \cdots, f_k$ are functions defining the germ $(V, 0)$, then $(v, \overline{\text{grad}}(f_1), \ldots, \overline{\text{grad}}(f_k))$ determines a trivialization of the bundle $T\mathbb{C}^{n+k}|_{\partial\mathbf{F}}$, and the corresponding relative Chern class of degree n equals the GSV index of v. This interpretation of the GSV index as a Chern class is closely related to the *virtual index* studied in Chap. 5.

3.7.3 On the Normal Bundle of Holomorphic Singular Foliations

An important problem in geometry, studied by several authors in various contexts, is that of extendability of vector bundles. In this section we show how the theory of indices of vector fields developed in this chapter can be used to investigate this problem for the normal bundle of a foliation.

Let $(V, 0)$ be an ICIS of dimension $n \geq 2$, and let v a holomorphic vector field on V, singular only at 0. This vector field defines a 1-dimensional holomorphic foliation $\mathcal{F}$ on V singular at 0. On $V \setminus \{0\}$, we have the tangent bundle $T\mathcal{F}$ to the foliation and we define the normal bundle $\nu(\mathcal{F})$ to be the quotient $T(V \setminus \{0\})/T\mathcal{F}$.

The following result provides a topological obstruction for the extendability of this bundle.

Corollary 3.7.1. *Let $(V, 0)$ and $\mathcal{F}$ be as above. Then the normal bundle to $\mathcal{F}$ in $V \setminus \{0\}$ extends to 0 as a (continuous or smooth) vector bundle if and only if the GSV index of v is a multiple of $(n-1)!$.*

An example for which this condition is not satisfied is the one of a linear vector field on $V = \mathbb{C}^n$, $n > 2$, since in this case the index is 1. When $n = 3$ one can actually say a little more:

Corollary 3.7.2. *Assume V has complex dimension 3. Let $\mathcal{F}$ be a holomorphic foliation on V spanned by a holomorphic vector field v, singular only at 0. Let $\nu(\mathcal{F})$ be the normal bundle of $\mathcal{F}$ in $V^* = V \setminus \{0\}$. Then the following conditions are equivalent:*

(1) *The GSV index of v at 0 is even.*

(2) *The bundle $\nu(\mathcal{F})$ admits a nowhere-zero C^∞ section.*

(3) *The bundle $\nu(\mathcal{F})$ is C^∞ trivial and therefore extends to a bundle over V.*

The first corollary above is obvious from Theorem 3.7.3. We notice only that if the normal bundle $\nu(\mathcal{F})$ on V^* is trivial, then it is isomorphic to $V^* \times$

$\mathbb{C}^{n-1}$ and therefore extends to V as the trivial bundle $V \times \mathbb{C}^{n-1}$. Conversely, if the bundle $\nu(\mathcal{F})$ on V^* extends to 0, then it is necessarily trivial at 0, since all bundles are locally trivial.

Concerning the second corollary, the equivalence between statements (1) and (3) is immediate from Theorem 3.7.3, and it is obvious that (3) implies (2), so we only must prove that (2) implies (3). Let ξ be a never-zero continuous section of the normal bundle $\nu(\mathcal{F})$. This spans a 1-dimensional continuous complex line sub-bundle $\mathcal{L}$ of $\nu(\mathcal{F})$. The bundle $\mathcal{L}$ is trivial iff $\nu(\mathcal{F})$ is trivial. But the link $\mathbf{K}$ is 2-connected, by [121]. Hence every complex line bundle over $\mathbf{K}$ is trivial.

Let us give an example. Let V be a hypersurface in $\mathbb{C}^{2n}$ defined by some function $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}, 0)$. Then the Hamiltonian vector field

$$\widetilde{\zeta} = \Big(\frac{\partial f}{\partial z_2}, -\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_4}, -\frac{\partial f}{\partial z_3}, \cdots, \frac{\partial f}{\partial z_{2n}}, -\frac{\partial f}{\partial z_{2n-1}}\Big)$$

defined as in Example 3.3.1 has GSV index 0. Thus, by 3.7.3, the normal bundle of the holomorphic foliation that v spans on V^* is topologically trivial.

Chapter 4
Indices of Vector Fields on Real Analytic Varieties

Abstract In the previous chapters we focused on indices of vector fields on complex analytic varieties. The real analytic setting also has its own interest, and that is the subject of this chapter. The following presentation follows the discussion by M. Aguilar, J. Seade and A. Verjovsky in [6] (see also [49]). We describe indices analogous to the GSV and Schwartz indices for vector fields on real analytic singular varieties. In this setting the GSV index is an integer if the singular variety V is odd-dimensional, but it is defined only modulo 2 if the dimension of V is even.

The Schwartz and the GSV indices are defined, respectively, in Sects. 1 and 2; there we show that the Schwartz index classifies the homotopy classes of vector fields near an isolated singularity. Section 3 provides a geometric interpretation of the GSV index in the real analytic setting.

The information we get is related to previous work by M. Kervaire about the *curvatura integra* of manifolds, and this is the subject we explore in Sect. 4. Finally, in Sect. 5 we look at the relation of these indices with other invariants of real analytic singularity germs studied previously by C. T. C. Wall and others. This yields to an extension of the concept of Milnor number for real analytic map-germs with isolated singularities which may not be algebraically isolated.

We note that there are some related works such as [9, 10, 49, 69, 70].

Since in this chapter we consider only real analytic varieties and functions, for simplicity, we will denote the dimensions here by m, n... instead of m', n'..., as in the rest of the book.

4.1 The Schwartz Index on Real Analytic Varieties

Let $(V, 0)$ be the germ at 0 of an irreducible, pure dimensional real analytic variety of dimension n in $\mathbb{R}^{n+k}$ with an isolated singularity at 0. We denote by V_{reg} its regular part: $V_{\text{reg}} := V \setminus \{0\}$. As before, a continuous (smooth or analytic) vector field on V means the restriction v to V of a continuous (smooth or analytic) vector field on a neighborhood of V in $\mathbb{R}^{n+k}$ which

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is tangent to V_{reg}. We want to define the Schwartz index of v at 0. As on complex varieties, this index measures how far the vector field is from being radial. We notice that as in the complex case, we may drop the condition of having an isolated singularity and assume only that the singular set of V is compact, see [6], but we restrict to the isolated singularity case for simplicity.

Assume first that the link $\mathbf{K}$ of V is connected. Let v_{rad} be a radial vector field at 0, *i.e.*, v_{rad} is transverse, outwards-pointing to the intersection of V with every sufficiently small sphere $\mathbb{S}_\varepsilon$ centered at 0. Define the difference between v and v_{rad} at 0 as before: consider small spheres $\mathbb{S}_\varepsilon$, $\mathbb{S}_{\varepsilon'}$; $\varepsilon > \varepsilon' > 0$, and let w be a vector field on the cylinder X in V bounded by the links $\mathbf{K}_\varepsilon = \mathbb{S}_\varepsilon \cap V$ and $\mathbf{K}_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap V$, such that w has finitely many singularities in the interior of X, it restricts to v on $\mathbf{K}_\varepsilon$ and to v_{rad} on $\mathbf{K}_{\varepsilon'}$. *The difference* of v and v_{rad} is defined by:

$$d(v, v_{\text{rad}}) = \text{Ind}_{\text{PH}}(w, X)\,,$$

the Poincaré–Hopf index of w on X. Then define *the Schwartz index* of v at $0 \in V$ to be:

$$\text{Ind}_{\text{Sch}}(v, 0; V) = 1 + d(v, v_{\text{rad}}).$$

In particular v_{rad} has index 1 (which equals the Poincaré–Hopf index of its radial extension to a ball in $\mathbb{R}^{n+k}$).

If the analytic variety V is pure-dimensional, with an isolated singularity at 0, but it has several irreducible components $V_1, \cdots, V_r$, and v is as above, a vector field on a neighborhood of 0 in $\mathbb{R}^{n+k}$, tangent to V_{reg}, one can define the Schwartz multi-index $\text{Ind}^{\text{multi}}_{\text{Sch}}(v, 0)$:

$$\text{Ind}^{\text{multi}}_{\text{Sch}}(v, 0) \;:=\; (\text{Ind}_{\text{Sch}}(v_1, 0), \ldots, \text{Ind}_{\text{Sch}}(v_r, 0)).$$

Notice that if V is a compact, oriented, pure dimensional, irreducible real analytic variety and v is a vector field on V with only finitely many singularities (or zeroes) $x_1, \ldots, x_s$ on V_{reg}, one has at each x_i the local Schwartz index of v. The *Total Schwartz index* $\text{Ind}_{\text{Sch}}(v, V)$ of v in V is defined in the obvious way and Theorem 2.1.1 generalizes to this setting:

$$\text{Ind}_{\text{Sch}}(v, V) = \chi(V).$$

A similar remark holds for the multi-index if V has several irreducible components.

Let us now show that the Schwartz index classifies the homotopy classes of continuous vector fields on V. Notice that the same arguments work in the complex analytic case when the variety has an isolated singularity.

Definition 4.1.1. Let $(V, 0)$ be a real analytic germ as above, and let v and w be vector fields on V that vanish only at the singular point $0 \in V$. We say that v and w are *homotopic* if there exists a continuous 1-parameter family

w_t of vector fields on V, $t \in [0,1]$, such that $w_0 = v$, $w_1 = w$ and for each t the vector field w_t vanishes only at $0 \in V$. We denote by $\Theta(V,0)$ the set of homotopy classes of such vector fields.

We remark that it is important to demand that the homotopies be through vector fields that vanish only at 0, otherwise all vector fields are homotopic.

Proposition 4.1.1. *Let $\mathbf{K}_1, \ldots, \mathbf{K}_r$ be the connected components of the link $\mathbf{K}$. Then the set $\Theta(V,0)$ corresponds bijectively with $\oplus_{i=1}^r \mathbb{Z}$ and a bijection between $\Theta(V,0)$ and $\oplus_{i=1}^r \mathbb{Z}$ is given by the Schwartz multi-index:*

$$v \mapsto \mathrm{Ind}_{\mathrm{Sch}}^{\mathrm{multi}}(v,0) := (\mathrm{Ind}_{\mathrm{Sch}}(v_1,0), \ldots, \mathrm{Ind}_{\mathrm{Sch}}(v_r,0)).$$

Proof. Let $\Theta(\mathbf{K})$ be the set of homotopy classes of never vanishing vector fields tangent to V on $\mathbf{K}$. By [121], V is the cone over the link $\mathbf{K}$. Hence, there is a canonical bijection between $\Theta(V,0)$ and $\Theta(\mathbf{K})$. We show that $\Theta(\mathbf{K})$ is classified by the Schwartz multi-index, and that for each connected component of $\mathbf{K}$ there is exactly one homotopy class of tangent vector fields corresponding to each integer, which proves the statement. It is obviously enough to consider the case where $\mathbf{K}$ is connected.

Recall that given vector fields v and v' on V, never-zero on $\mathbf{K}$ (assumed to be connected), the difference is well defined as in Chap. 1. It is clear that if v and v' are homotopic, then their difference is zero, so they have the same Schwartz index. Conversely, if they have the same Schwartz index, then their difference is 0, hence they are homotopic. Thus the homotopy classes of such vector fields are classified by their Schwartz index. It remains to see that there are vector fields of all possible Schwartz indices, but this is easy: let $\varepsilon > \varepsilon' > 0$ be sufficiently small, let $\mathbf{K}_\varepsilon$, $\mathbf{K}'_\varepsilon$ be links of these radius and $X \subset V$ the cylinder bounded by $\mathbf{K}_\varepsilon$, $\mathbf{K}'_\varepsilon$. Put on $\mathbf{K}'_\varepsilon$ a vector field v of some given Schwartz index, say $I(v)$; now choose in the interior of X a small disk $\mathbb{D}$ and put on it a vector field v' of some index I'. By [153] we can extend v and v' to a vector field w on X, non singular on $\mathbf{K}_\varepsilon$ and with no other singularity but that on $\mathbb{D}$. By construction, the Schwartz index of w is $I(v) + I'$.

4.2 The GSV Index on Real Analytic Varieties

We now consider the analogous of the GSV index for vector fields on real analytic germs. For this we first need to define the *index (or degree)* of a map from a smooth $(n-1)$-manifold into the Stiefel manifold $V_{k+1,n+k}$ of orthonormal $(k+1)$-frames in $\mathbb{R}^{n+k}$, with $k > 0$, $n > 1$. We recall that $V_{k+1,n+k}$ is an $(n-1)$-sphere bundle over $V_{k,n+k}$ and that, by [164], one has a canonical embedding $\gamma : \mathbb{S}^{n-1} \to V_{k+1,n+k}$, which is the fiber over $(e_{n+1}, \ldots, e_{n+k}) \in V_{k,n+k}$, where $e_1, \ldots, e_{n+k}$ is the canonical basis of $\mathbb{R}^{n+k}$.

The homotopy class of γ is a generator of $\pi_{n-1}(V_{k+1,n+k})$, which is isomorphic to $\mathbb{Z}$ if n is odd or to $\mathbb{Z}/2\mathbb{Z}$ if n is even, since $k > 0$, see [153].

By the Universal Coefficient Theorem, we have an isomorphism

$$H^{n-1}(V_{k+1,n+k};\pi_{n-1}(V_{k+1,n+k})) \xrightarrow{\alpha} \mathrm{Hom}(H_{n-1}(V_{k+1,n+k};\mathbb{Z}), \pi_{n-1}(V_{k+1,n+k})).$$

Let $h\colon \pi_{n-1}(V_{k+1,n+k}) \to H_{n-1}(V_{k+1,n+k};\mathbb{Z})$ be the Hurewicz homomorphism. Since $V_{k+1,n+k}$ is $(n-2)$-connected, h is an isomorphism and it is given by $h[\gamma] = \gamma_*[\mathbb{S}^{n-1}]$, where $[\mathbb{S}^{n-1}]$ is the fundamental class. Hence,

$$u := \alpha^{-1}(h^{-1}) \in H^{n-1}(V_{k+1,n+k};\pi_{n-1}(V_{k+1,n+k})),$$

is a characteristic element determined by the equality $\langle u, \gamma_*[\mathbb{S}^{n-1}]\rangle = [\gamma]$, where $\langle , \rangle$ denotes the Kronecker product. The generator $[\gamma] \in \pi_{n-1}(V_{k+1,n+k}))$ gives the characteristic element u and an isomorphism from $\pi_{n-1}(V_{k+1,n+k}))$ to $\mathbb{Z}$ or to $\mathbb{Z}/2\mathbb{Z}$.

Definition 4.2.1. Let N be an $(n-1)$-dimensional, closed oriented manifold (maybe not connected) and let $g\colon N \to V_{k+1,n+k}$ be a map, $n > 1$, $k > 0$. We define the *degree* of g as follows:

(1) If n is odd, then $\deg(g) := \langle g^*(u), [N]\rangle = \langle u, g_*[N]\rangle \in \mathbb{Z}$, where $[N]$ is the fundamental class with integer coefficients.
(2) If n is even, then $\deg_2(g) := \langle g^*(u), [N]_2\rangle = \langle u, g_*[N]_2\rangle \in \mathbb{Z}/2\mathbb{Z}$, where $[N]_2$ is the fundamental class with mod 2 coefficients.

Let us now denote by $(V, 0)$ the germ of a geometric complete intersection with an isolated singularity at 0. This means that V is defined by a real analytic map

$$f := (f_1, f_2, \dots, f_k) : U \subset \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k \quad n > 1, k > 0,$$

where U is an open neighborhood of 0 in $\mathbb{R}^{n+k}$, such that the gradient vector fields $(\mathrm{grad_1}, \dots, \mathrm{grad_k})$ of the f_i are linearly independent everywhere on $V \setminus \{0\}$, and they are of course normal to $V \setminus \{0\}$. Let v be a continuous vector field on V which is singular only at 0. Let us define now the index of v at 0 following [6]. Notice that up to normalization one has a continuous map,

$$\phi_v := (v, \mathrm{grad_1}, \dots, \mathrm{grad_k}) : \mathbf{K} \longrightarrow \mathrm{V_{k+1,n+k}},$$

where $V_{k+1,n+k}$ is the Stiefel manifold of orthonormal $(k+1)$-frames in $\mathbb{R}^{n+k}$ and $\mathbf{K}$ is the link of 0 in V.

Definition 4.2.2. If n is odd, then *the (real) GSV index* of v at 0 is the integer defined by

$$\mathrm{Ind_{GSV}}(v, 0) = \deg(\phi_v).$$

If n is even, then *the (real) mod(2) GSV index* of v at 0 is the integer modulo 2 defined by

$$\mathrm{Ind}_{\mathrm{GSV}_2}(v,0) = \deg_2(\phi_v).$$

Definition 4.2.3. Let $\mathbf{K}_1, \ldots, \mathbf{K}_r$ be the connected components of $\mathbf{K}$. If $n = \dim V$ is odd, then we define the *multi-index* $\mathrm{Ind}^{\mathrm{multi}}_{\mathrm{GSV}}(v)$ of a vector field v on V by:

$$\mathrm{Ind}^{\mathrm{multi}}_{\mathrm{GSV}}(v,0) = (\deg(\phi_{v_1}), \ldots, \deg(\phi_{v_r})),$$

where ϕ_{v_i} is the restriction of ϕ_v to the component $\mathbf{K}_i$.

Similar considerations apply to the mod-2 index when n is even.

The following proposition gives a geometric interpretation of the multi-indices:

Proposition 4.2.1. *Let $\gamma : \mathbb{S}^{n-1} \to V_{k+1,n+k}$ be the canonical embedding defined above, and let $\phi\colon \mathbf{K} \to V_{k+1,n+k}$ be a continuous map. Then there exists a map $\tilde{\phi}\colon \mathbf{K} \to \mathbb{S}^{n-1}$, unique up to homotopy, such that $\gamma \circ \tilde{\phi}$ is homotopic to ϕ. Furthermore, if n is odd, then for each connected component $\mathbf{K}_i$ of $\mathbf{K}$ the degree of $\tilde{\phi}$ restricted to $\mathbf{K}_i$ equals the index $\mathrm{Ind}(\phi_i)$ of $\phi_i := \phi|_{\mathbf{K}_i}$. If n is even, then the reduction modulo 2 of the degree of $\tilde{\phi}$ restricted to $\mathbf{K}_i$ equals the index $\mathrm{ind}_2(\phi_i)$ of $\phi_i := \phi|_{\mathbf{K}_i}$.*

In order to prove this proposition we prove first the following lemma, which is also used later.

Lemma 4.2.1. *Let $\mathbf{K}$ be as above, let $\mathbf{K}_1, \ldots, \mathbf{K}_r$ be the connected components of $\mathbf{K}$, and denote by g_i the restriction to $\mathbf{K}_i$ of a map $g\colon \mathbf{K} \to V_{k+1,n+k}$.*

(1) *If n is odd, then there is a bijection between the group of homotopy classes of maps $[\mathbf{K}, V_{k+1,n+k}]$ and $\oplus_{i=1}^r \mathbb{Z}$, given by $[g] \mapsto (\deg(g_1), \ldots, \deg(g_r))$.*
(2) *If n is even, then there is a bijection between $[\mathbf{K}, V_{k+1,n+k}]$ and $\oplus_{i=1}^r \mathbb{Z}/2\mathbb{Z}$, given by $[g] \mapsto (\deg_2(g_1), \ldots, \deg_2(g_r))$.*

Proof. Clearly $H^q(N; \pi_q(V_{k+1,n+k})) \cong 0$ and $H^{q+1}(N; \pi_{q+1}(V_{k+1,n+k})) \cong 0$, for all $q > n-1$. Therefore, by obstruction theory [153], Theorem 8.4.3, there is a bijection between $[N, V_{k+1,n+k}]$ and $H^{n-1}(N; \pi_{n-1}(V_{k+1,n+k}))$, given by $[g] \mapsto g^*(u)$, where u is as in 4.2.1 above. Now consider the following composition:

$$[N, V_{k+1,n+k}] \to H^{n-1}(N;R) \xrightarrow[\cong]{p} H_0(N;R) \xleftarrow[\cong]{\overset{r}{\underset{i=1}{\oplus}} j_i} \bigoplus_{i=1}^r H_0(N_i;R) \xrightarrow[\cong]{\overset{r}{\underset{i=1}{\oplus}} \varepsilon_i} \bigoplus_{i=1}^r R,$$

where p is Poincaré duality, $j_i\colon N_i \to N$ are the inclusions, and each ε_i is the augmentation. We take $R = \mathbb{Z}$ if n is odd and $R = \mathbb{Z}/2\mathbb{Z}$ if n is even. Thus one has a bijection

$$\zeta : \ [N, V_{k+1,n+k}] \longrightarrow \bigoplus_{i=1}^{r} R.$$

Since $p(a) = a \cap [N]$, we have that

$$\deg(g_i) := \langle g_i^*(u), [N_i] \rangle = \varepsilon_i (g_i^*(u) \cap [N_i]) = \varepsilon_i p g_i^*(u).$$

A straightforward calculation shows that $\zeta[g] = (\deg(g_1), \ldots, \deg(g_r))$ if n is odd and that $\zeta[g] = (\deg_2(g_1), \ldots, \deg_2(g_r))$ if n is even.

Proof of Proposition 4.2.1: Consider first the case n odd. From the proof of the previous lemma we know that we have a bijection

$$s : [\mathbf{K}, V_{k+1,n+k}] \longrightarrow H^{n-1}(\mathbf{K}; \mathbb{Z}).$$

By obstruction theory we also have a bijection

$$t : [\mathbf{K}, \mathbb{S}^{n-1}] \longrightarrow H^{n-1}(\mathbf{K}; \mathbb{Z}).$$

Recall that γ induces a function

$$\gamma_* : [\mathbf{K}, \mathbb{S}^{n-1}] \longrightarrow [\mathbf{K}, V_{k+1,n+k}] \, ,$$

given by $\gamma_*[f] = [\gamma \circ f]$. Then one can check that $t = s \circ \gamma_*$. Thus γ_* is a bijection, as stated. The second statement now follows from the definition of the index and the fact that $f_*[\mathbf{K}] = \deg(f)[\mathbb{S}^{n-1}]$.

The case n be even is proved in the same way, but now using the bijection:

$$[\mathbf{K}, V_{k+1,n+k}] \longleftrightarrow H^{n-1}(\mathbf{K}; \mathbb{Z}/2\mathbb{Z}) \qquad \square$$

The following result is an immediate consequence of 4.1.2 and 4.2.1.

Proposition 4.2.2. *Let* $\mathbf{K}_1, \ldots, \mathbf{K}_r$ *be the connected components of the link* $\mathbf{K}$.

(1) *If n is odd, then there is a bijection between $\Theta(V, 0)$ and $\oplus_{i=1}^{r} \mathbb{Z}$, given by*

$$[v] \mapsto \mathrm{Ind}_{\mathrm{GSV}}^{\mathrm{multi}}(v, 0) := (\mathrm{Ind}_{\mathrm{GSV}}(v_1, 0), \ldots, \mathrm{Ind}_{\mathrm{GSV}}(v_r, 0)).$$

(2) *If n is even, then there is a surjection from $\Theta(V, 0)$ to $\oplus_{i=1}^{r} \mathbb{Z}/2\mathbb{Z}$ given by*

$$[v] \mapsto \mathrm{Ind}_{\mathrm{GSV}_2}^{\mathrm{multi}}(v, 0) := (\mathrm{Ind}_{\mathrm{GSV}_2}(v_1, 0), \ldots, \mathrm{Ind}_{\mathrm{GSV}_2}(v_r, 0)).$$

4.3 A Geometric Interpretation of the GSV Index

Let U be an open neighborhood of 0 in $\mathbb{R}^{n+k}$. Consider the map

$$f := (f_1, f_2, \dots, f_k) : (U, 0) \longrightarrow (\mathbb{R}^k, 0), \quad n > 1,\ k > 0,$$

that defines the isolated complete intersection singularity $(V, 0)$, and consider the fibers $f^{-1}(t)$ for t near 0. If t is a regular value of f, we call $f^{-1}(t)$ a *nonsingular level surface* of f. Its intersection $\mathbf{F}_t := f^{-1}(t) \cap \mathbb{D}_\varepsilon$ with a small disc $\mathbb{D}_\varepsilon$ around $0 \in U \subset \mathbb{R}^{n+k}$ is a *nonsingular fiber* of f. We remark that we do not have in general a fibration as in the case of complex singularities studied in [79, 116, 121]. However, by hypothesis 0 is an isolated singularity in V, hence the Jacobian matrix $Df(x)$ has rank k at each $x \in V \setminus \{0\}$. Thus there exist $\varepsilon > \varepsilon' > 0$ and $\delta > 0$ sufficiently small with respect to ε', such that $Df(x)$ has rank k on the set Ω of all $x \in U$ such that $\varepsilon > ||x|| > \varepsilon'$ and $f(x) \in \Delta_\delta$, where Δ_δ is a small ball in $\mathbb{R}^k$ centered at 0. By Ehresmann fibration lemma, this implies that the restriction of f to Ω is the projection map of a locally trivial fiber bundle over Δ_δ. The Transversal Isotopy Theorem [4] implies that we can move $V \cap \Omega$ by an ambient isotopy and take it into $\mathbf{F}_t \cap \Omega$, where $\mathbf{F}_t$ is a nonsingular fiber of f. This carries the vector field v to a nowhere-zero vector field on $\mathbf{F}_t \cap \Omega$, provided $\varepsilon' >> |t| > 0$. Thus one has the following lemma:

Lemma 4.3.1. *There exists an ambient isotopy in Ω that carries $V \cap \Omega$ into $\mathbf{F}_t \cap \Omega$ and takes v into a nonsingular vector field, also denoted by v, defined in a neighborhood of the boundary $\partial\mathbf{F}_t$, and this boundary is isotopic to the link* $\mathbf{K}$.

By Theorem 1.1.2 one can extend v to a vector field on the whole fiber $\mathbf{F}_t$ with only one singular point, say p, in the interior of $\mathbf{F}_t$. The local Poincaré–Hopf index of v at p is independent on the way we extend v to the interior of $\mathbf{F}_t$, and this number is the Poincaré–Hopf index of v in $\mathbf{F}_t$, $\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t)$. We note that the gradient vector fields $(\mathrm{grad_1}, \dots, \mathrm{grad_k})$ are linearly independent on all of $\mathbf{F}_t$ because t is a regular value of f. Thus, if we let D_ε be a small disk in $\mathbf{F}_t$ centered at p and ∂D_ε is its boundary, then the above map

$$\phi_v = (v, \mathrm{grad_1}, \dots, \mathrm{grad_k}) : \mathbf{K} \longrightarrow \mathrm{V_{k+1,n+k}},$$

extends to a continuous map from all of $\mathbf{F}_t - D_\varepsilon$ into $V_{k+1,n+k}$ which factors through $\partial D_\varepsilon \cong \mathbb{S}^{n-1}$, up to homotopy. Thus the index of ϕ_v equals the index of the corresponding map ϕ_v from a small sphere around p into $V_{k+1,n+k}$, and the latter equals $\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t)$ by definition. Hence one has:

Theorem 4.3.1. *Up to isotopy, the vector field v can be regarded as a vector field defined and never-zero on a neighborhood of the boundary of the nonsingular fiber $\mathbf{F}_t$ and one has:*

(1) *If* n *is odd, then* $\mathrm{Ind}_{\mathrm{GSV}}(v, 0)$ *equals the Poincaré–Hopf index of* v *on* $\mathbf{F}_t$, $\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t)$.

(2) *If* n *is even, then the mod*(2)*-GSV index of* v, $\mathrm{Ind}_{\mathrm{GSV}_2}(v, 0)$, *is the reduction of* $\mathrm{Ind}_{\mathrm{PH}}(v, \mathbf{F}_t)$ *modulo* 2.

In other words, if the dimension $(n-1)$ of $\mathbf{K}$ is even then the *GSV*-index of v is the number of zeroes of an extension of v to the nonsingular fiber $\mathbf{F}_t$, counted with their local indices. If the dimension of $\mathbf{K}$ is odd, then the reduction modulo 2 of the number of zeroes of v in $\mathbf{F}_t$, counted with local indices, equals $\mathrm{Ind}_{\mathrm{GSV}_2}(v, 0)$. Notice that this theorem and its proof are reminiscent of [89], Lemma 2.

Remark 4.3.1. Recall that in the complex analytic case one has the *GSV* index of the previous chapter, which is an integer, and its mod 2-reduction coincides with the index defined in 4.2.3. In general, for real analytic mappings as above, if n is even (as for instance if the singularity germs is actually complex analytic), one only has a GSV-index defined modulo 2. One might be tempted to define an index over the integers as in Chap. 3, *i.e.*, by looking at the number of zeroes of the vector field on a nearby fiber. The problem is that for n even, the number one gets depends on the choice of fiber and one can only get a well defined index modulo 2 in this way (see Sect. 3 of Chap. 7 for a related discussion).

Of course this discussion is very much related to the important problem of computing the Euler–Poincaré characteristic of the regular fibers of analytic maps, and there is a vast literature about that topic.

When $k = 1$, the topology of the fibers may change as we pass from $t > 0$ to $t < 0$, and one can speak of *right* and *left GSV-indices.* When n is even these two indices coincide, but for n odd, in general they coincide only modulo 2.

4.4 Topological Invariants and Curvatura Integra

If Z is an oriented $(n-1)$-dimensional closed submanifold of the Euclidean space $\mathbb{R}^n$, then its normal bundle is necessarily trivial. If ν is a section of the normal bundle "pointing outwards" everywhere, then ν determines a map from Z into the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$, the *Gauss map*, whose degree is the *curvatura integra* of Z. Hopf's generalization in [85] of the theorem of Gauss states that if $n-1$ is even then its *curvatura integra* is half the Euler–Poincaré characteristic of Z, independently of the embedding. This theorem was generalized by Kervaire in [89] to submanifolds of Euclidean space of arbitrary dimensions, but embedded with trivial normal bundle. Let Z^{n-1} be an $(n-1)$-manifold embedded in $\mathbb{R}^{n+k}$ and let $v^{(k+1)} := (u_0, \ldots, u_k)$ be a trivialization of its normal bundle. We call $v^{(k+1)}$ a *framing* of Z in $\mathbb{R}^{n+k}$. The framing defines a continuous map:

$$\phi : Z \longrightarrow V_{k+1,n+k}.$$

This map induces a homomorphism $\phi_*\colon H_{n-1}(Z;\mathbb{Z}) \to H_{n-1}(V_{k+1,n+k};\mathbb{Z})$, which takes the fundamental class $[Z]$ into $c_Z[g]$, where c_Z is an integer if $(n-1)$ is even or an integer modulo 2 when $(n-1)$ is odd and $[g]$ is the generator of $H_{n-1}(V_{k+1,n+k};\mathbb{Z})$. In either case Kervaire [89] calls c_Z the *(generalized) curvatura integra* of Z, and proved that if $(n-1)$ is even, then c_Z is half the Euler–Poincaré characteristic of Z, independently of the embedding of Z in $\mathbb{R}^{n+k}$. For $(n-1)$ odd the similar statement is false in general but it is true if Z bounds a stably-parallelizable manifold and if we replace "half the Euler–Poincaré characteristic of Z" by the semi-characteristic of Z, as shown below (following [6, 89]).

Definition 4.4.1. [89] Let Y be a manifold of dimension $2p-1$. The *semi-characteristic* of Y with respect to a coefficient field $\mathbb{Z}/2\mathbb{Z}$ is

$$\chi_{\frac{1}{2}}(Y) = \sum_{i=0}^{p-1} \mathrm{rank} H_i(Y;\mathbb{Z}/2\mathbb{Z}).$$

The following result is of interest in itself and we use it below. This is mentioned in [89] with an outline of its proof, which is given in detail in [6].

Proposition 4.4.1. *Let N be a compact, stably parallelizable manifold with boundary ∂N. If the dimension n of N is even, then*

$$\chi(N) = \chi_{\frac{1}{2}}(\partial N) \mod 2.$$

Consider now the germ $(V,0)$ of a geometric complete intersection with an isolated singularity at 0, defined by a real analytic map

$$f := (f_1, f_2, \ldots, f_k) : U \subset \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k, \quad n > 1,\ k > 0,$$

where U is an open neighborhood of 0 in $\mathbb{R}^{n+k}$. Let $\mathbf{K}$ be the link of 0 in V, which may not be connected; $\mathbf{K}$ has dimension $n-1$.

Lemma 4.4.1. *Let v_{rad} be a radial, outwards-pointing vector field on V.*

(1) *If n is odd, then the GSV-index of v_{rad} equals the curvatura integra of* $\mathbf{K}$. *Thus:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0) = \frac{1}{2}\chi(\mathbf{K}).$$

(2) *If n is even, then the mod(2) GSV-index of v_{rad} equals the curvatura integra of* $\mathbf{K}$.

Proof. Let v_{rad} be as above. Up to homotopy, we can think of v_{rad} as being the unit outwards normal field of $\mathbf{K}$ in V and let $(\mathrm{grad_1}, \ldots, \mathrm{grad_k})$ be as before, the gradient vector fields of $(f_1, f_2, \ldots, f_k)$. Then v_{rad} and $(\mathrm{grad_1}, \ldots, \mathrm{grad_k})$ determine a continuous map from $\mathbf{K}$ into the Stiefel manifold $V_{k+1,n+k}$. By definition, the index of this map is the GSV-index of v_{rad}. We recall that $c_{\mathbf{K}}$ is defined by the equality $\phi_*[\mathbf{K}] = c_{\mathbf{K}}[g]$, where $\phi\colon \mathbf{K} \to V_{k+1,n+k}$ is the map determined by the framing. But we have that $[g] = \gamma_*[\mathbb{S}^{n-1}]$, where $\gamma\colon \mathbb{S}^{n-1} \to V_{k+1,n+k}$ is the generator of $\pi_{n-1}(V_{k+1,n+k})$ mentioned in Sect. 4.2. Hence $\phi_*[\mathbf{K}] = c_{\mathbf{K}}\gamma_*[\mathbb{S}^{n-m1}]$. By 4.2.1,

$$\deg(\phi) = \langle \phi^*(u), [\mathbf{K}]\rangle = \langle u, \phi_*[\mathbf{K}]\rangle,$$

and $\langle u, \gamma_*[\mathbb{S}^{n-1}]\rangle = [\gamma]$. Under the identification of $\pi_{n-1}(V_{k+1,n+k})$ with $\mathbb{Z}$, $[\gamma]$ corresponds to 1, so that $\langle u, \gamma_*[\mathbb{S}^{n-1}]\rangle = 1$. Therefore

$$\deg(\phi) = \langle u, \phi_*[\mathbf{K}]\rangle = \langle u, c_{\mathbf{K}}\gamma_*[\mathbb{S}^{n-1}]\rangle = c_{\mathbf{K}}\langle u, \gamma_*[\mathbb{S}^{n-1}]\rangle = c_{\mathbf{K}}.$$

Similarly for $\mathrm{ind}_2(\phi)$. Moreover, the index of the corresponding map into $V_{k+1,n+k}$ does not change if we replace v_{rad} by any other vector field on V which is also transverse to $\mathbf{K}$ and is *radial, outwards pointing.* The result now follows from [89, Theorem VI].

We have the following theorem:

Theorem 4.4.1. *Let* v_{rad} *be a radial, outwards-pointing vector field on* V*, and let* $\mathbf{F}_t$ *be a nonsingular fiber of* f*.*

(1) *If* n *is odd, then the* GSV*-index of* v_{rad} *equals the curvatura integra of* $\mathbf{K}$ *and we have:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0) = \frac{1}{2}\chi(\mathbf{K}) = \chi(\mathbf{F}_t).$$

(2) *If* n *is even, then the mod(2)* GSV*-index of* v_{rad} *equals the curvatura integra of* $\mathbf{K}$ *and we have:*

$$\mathrm{Ind}_{\mathrm{GSV}_2}(v_{\mathrm{rad}}, 0) = \chi_{\frac{1}{2}}(\mathbf{K}) = \chi(\mathbf{F}_t) \mod 2.$$

Proof. Statement (1) follows from the previous lemma and the fact that if X is a compact, odd-dimensional manifold with boundary ∂X, then one always has $\chi(\partial X) = 2\chi(X)$. Let us now assume that n is even. Also by the lemma above we have that $\mathrm{Ind}_{\mathrm{GSV}_2}(v_{\mathrm{rad}}, 0)$ is the curvatura integra of $\mathbf{K}$. Thus, by [89], Lemma 2, we have

$$\mathrm{Ind}_{\mathrm{GSV}_2}(v_{\mathrm{rad}}, 0) = \chi(\mathbf{F}_t) \bmod 2.$$

Hence, to complete the proof of (2) we must show that $\chi(\mathbf{F}_t) = \chi_{\frac{1}{2}}(\mathbf{K})$ *mod* 2, but this follows from 4.4.1.

The following is an immediate consequence of the theorem above.

Corollary 4.4.1. *Let* $(V_1, 0)$ *and* $(V_2, 0)$ *be* n*-dimensional, isolated, complete intersection germs in* $\mathbb{R}^{n+k}$*. Let* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be the corresponding links, and* *let* $\mathbf{F}_t^1$ *and* $\mathbf{F}_{t'}^2$ *be the corresponding nonsingular fibers. Assume that* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *are orientation-preserving homeomorphic. If* n *is odd, then*

$$\chi(\mathbf{F}_t^1) = \chi(\mathbf{F}_{t'}^2).$$

If n *is even, then*

$$\chi(\mathbf{F}_t^1) = \chi(\mathbf{F}_{t'}^2) \bmod 2.$$

4.5 Relation with the Milnor Number for Real Singularities

In the complex case, we know that if $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a holomorphic function with an isolated critical point at $0 \in \mathbb{C}^{n+1}$, then f determines a fiber bundle,

$$f : f^{-1}(\mathbb{S}_\delta^1) \cap \mathbb{B}_\varepsilon \longrightarrow \mathbb{S}_\delta^1 \subset \mathbb{C},$$

where $\mathbb{S}_\delta^1 \subset \mathbb{C}$ is a small sphere and $\mathbb{B}_\varepsilon \subset \mathbb{C}^{n+1}$ is a small ball. This is equivalent to Milnor's fibration [121]. The fiber $\mathbf{F} = \mathbf{F}_t$ is called *the Milnor fiber* of f and it has the homotopy type of a wedge of n-spheres. The number $\mu = \mu(f)$ of spheres in this wedge is *the Milnor number* of f. These definitions and results extend to complete intersection germs by [79], and in fact, to some extent, to "isolated singularities" in general, by [102]. By [79,121], for isolated complete intersection singularity germs one has,

$$(-1)^n \mu = \chi(\mathbf{F}) - 1.$$

The Milnor number of such a singularity is a topological invariant in the sense that if f, g define germs as above and if there exists an orientation preserving local homeomorphism h in the domain, such that $f = g \circ h$, then $\mu(f) = \mu(g)$.

We recall that for vector fields on (isolated complete intersection) complex singularities the GSV index is an integer and one has

$$\mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, V) = \chi(\mathbf{F}),$$

where v_{rad} is the radial vector field on the isolated complete intersection singularity $(V, 0)$.

For real singularities a Milnor fibration does not exist in general. However, assume

$$f = (f_1, \ldots, f_k)\colon (\mathbb{R}^{n+k}, 0) \longrightarrow (\mathbb{R}^k, 0),$$

is an irreducible, complete intersection such that its complexification

$$f^{\mathbb{C}} = (f_1^{\mathbb{C}}, \ldots, f_k^{\mathbb{C}})\colon (\mathbb{C}^{n+k}, 0) \longrightarrow (\mathbb{C}^k, 0),$$

has an isolated singularity at $0 \in V_{\mathbb{C}}$, where $V_{\mathbb{C}} = \{z \in \mathbb{C}^{n+k} | f^{\mathbb{C}}(z) = 0\}$. In this case one says that the singularity of f at 0 is *algebraically isolated.* For these singularities, C.T.C. Wall introduced in [170] the following invariant,

$$\psi(\mathbf{K}) = \frac{1}{2}\beta(\mathbf{K}),$$

where $\mathbf{K}$ is the link of 0 in $V = f^{-1}(0)$ and

$$\beta(\mathbf{K}) = \dim_{\mathbb{Z}_2} H^*(\mathbf{K}, \mathbb{Z}_2).$$

He proved that modulo 2, $\psi(\mathbf{K})$ coincides with $\mu(f^{\mathbb{C}}) + 1$ where $\mu(f^{\mathbb{C}})$ is the Milnor number of $f^{\mathbb{C}}$ at 0. Hence the Milnor number $\mu(f^{\mathbb{C}})$ of the complexification, modulo 2, is a topological invariant of f. One also has the map,

$$f\colon f^{-1}(\mathbb{S}_\delta^{k-1}) \cap D_\varepsilon \longrightarrow \mathbb{S}_\delta^{k-1} \subset \mathbb{R}^k,$$

where $\mathbb{S}_\delta^{k-1}$ is a small sphere and $D_\varepsilon \subset \mathbb{R}^{n+k}$ is a small ball, as in the complex case. This is not a fiber bundle in general, however by the previous results we have that for regular values sufficiently near $0 \in \mathbb{R}^k$, the Euler–Poincaré characteristic of the fibers is independent of t if n is odd (modulo 2 for n even). Hence, given f as above, one has a well defined "*real Milnor number*" of f,

$$\mu_{\mathbb{R}}(f) = \chi(\mathbf{F}_t) - 1,$$

where this equality is modulo 2 if n is even.

Theorem 4.5.1. *Let v be a vector field on V singular only at $0 \in \mathbb{R}^{n+k}$. Then:*

(1) *The number*

$$\mu_{\mathbb{R}}(f) = (-1)^{n+1}\{\mathrm{Ind}_{\mathrm{GSV}}(v, 0; V) - \mathrm{Ind}_{\mathrm{Sch}}(v, 0; V)\} = (-1)^{(n+1)}(\chi(\mathbf{F}_t) - 1)$$

is independent of the choice of the vector field v (modulo 2 if n is even).

(2) *This is a topological invariant of f, i.e., if g is another complete intersection germ and h is a local orientation preserving homeomorphism of $\mathbb{R}^{n+k}$ such that $g = f \circ h$, then $\mu_{\mathbb{R}}(g) = \mu_{\mathbb{R}}(f)$ (modulo 2 if n is even).*

(3) *Let v_{rad} be a radial vector field on V and let $\psi(\mathbf{K})$ be Wall's invariant. If n is odd, then :*

$$\psi(\mathbf{K}) = \mathrm{Ind}_{\mathrm{GSV}}(v_{\mathrm{rad}}, 0) = \frac{1}{2}\chi(\mathbf{K}) \mod 2,$$

If n is even, then:

$$\psi(\mathbf{K}) = \mathrm{Ind}_{\mathrm{GSV}_2}(v_{\mathrm{rad}}, 0) = \chi_{\frac{1}{2}}(\mathbf{K}) \mod 2.$$

(4) *If the critical point of f is algebraically isolated, the real Milnor number $\mu_{\mathbb{R}}(f)$ modulo 2 coincides with the Milnor number $\mu(f^{\mathbb{C}})$ of the complexification.*

(5) *If W is a compact, oriented real analytic variety with isolated singularities $x_1, \ldots, x_r$, which are all complete intersection germs, and if v is a continuous vector field on W, singular at the x_i's and possibly at some smooth points of W, then the total GSV index of v is*

$$\mathrm{Ind}_{\mathrm{GSV}}(v, W) = \chi(W) + \sum_i \mu_{\mathbb{R}}(x_i),$$

if the dimension n of W is odd, and if n is even, then

$$\mathrm{Ind}_{\mathrm{GSV}_2}(v, W) \equiv \chi(W) + \sum_i \mu_{\mathbb{R}}(x_i) \bmod 2,$$

where the total GSV index is defined in the obvious way.

Proof. Statements (1), (2), and (5) are now obvious. To prove (3) we note that by definition:

$$\psi(\mathbf{K}) = \frac{1}{2}\beta(\mathbf{K}),$$

where

$$\beta(\mathbf{K}) = \dim_{\mathbb{Z}_2} H^*(\mathbf{K}, \mathbb{Z}_2).$$

Hence,

$$\psi(\mathbf{K}) \equiv \beta(\mathbf{F}_t) \mod 2,$$

by [170]. Thus the result follows because

$$\beta(\mathbf{F}_t) \equiv \chi(\mathbf{F}_t) \mod 2.$$

For statement (4) notice that by [170] we have that $\psi(\mathbf{K}) \equiv \mu(f^{\mathbb{C}}) + 1$ *mod* 2, and by definition $\psi(\mathbf{K}) \equiv \mu_{\mathbb{R}}(f) + 1$.

Chapter 5
The Virtual Index

Abstract The virtual index was first introduced in [111] by D. Lehmann, M. Soares and T. Suwa for holomorphic vector fields; the extension to continuous vector fields is immediate and has been done in [30, 31, 149]. If the variety has only isolated singularities, the virtual index and the GSV index coincide. The virtual index has several interesting features, as for instance that it is relatively easy to compute when the vector field we deal with is holomorphic, and also that it is defined for vector fields with singular set a compact set of arbitrary dimension.

In this chapter we introduce the virtual index in the context of singular varieties V which are local complete intersections defined by a section of a holomorphic vector bundle N over a complex manifold M (see Sect. 5.1 below). The virtual tangent bundle is then defined as $(TM - N)|_V$, where TM denotes the holomorphic tangent bundle of M.

One can think of the virtual index as being a localization of the top dimensional Chern class of the virtual tangent bundle, called virtual class, just as the local index of Poincaré–Hopf is a localization of the top Chern class of a manifold. The virtual index is in fact a residue which is the local contribution, relatively to a vector field v, of the top virtual class.

In Sect. 2, we show that Chern–Weil theory is very well adapted to this situation, in the framework of Čech-de Rham cohomology, and Sect. 3 is devoted to the study of residues in this context. The properties of the virtual index are detailed in the last sections of the chapter, in particular we prove an integral formula for the virtual index.

5.1 The Virtual Tangent Bundle of a Local Complete Intersection

In this section, the varieties we consider are "local complete intersections defined by a section." Namely, let V be a subvariety of dimension n in a complex manifold M of dimension $n + k$. We say that V is a local complete intersection defined by a section, if there exist a holomorphic vector bundle N

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of rank k over M and a holomorphic section s of N such that s is generically transverse to the zero section and that V is the zero set of s. Note that, in this case, the ideal sheaf of germs of holomorphic functions vanishing on V is generated by the local components of s and V is in fact a local complete intersection (cf. [165]), *i.e.*, each point of V has a neighborhood which is a complete intersection.

Let S be a compact set in V containing the singular set $\mathrm{Sing}(V)$ of V and with a finite number of connected components $(S_\lambda)_\lambda$. Suppose we have a nonvanishing C^∞ vector field v on $V \setminus S$ $(\subset V_{\mathrm{reg}})$.

If a component S_λ is in the regular part $V_{\mathrm{reg}} = V \setminus \mathrm{Sing}(V)$ of V, we saw in Chap. 1 that we can define the Poincaré–Hopf index $\mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda)$ of v at S_λ, which is the localization by v of the top Chern class of the tangent bundle TV_{reg} at S_λ. The question now is what to do if S_λ contains singular points of V, where there is no tangent bundle. The idea to define the virtual index is to make a similar "localization" using the vector field and using the fact that a variety V defined as above admits a *virtual tangent bundle*.

To define this bundle we notice that the restriction $N|_{V_{\mathrm{reg}}}$ coincides with the (holomorphic) normal bundle $N_{V_{\mathrm{reg}}}$ of V_{reg} in M so that we have an exact sequence.

$$0 \longrightarrow TV_{\mathrm{reg}} \longrightarrow TM|_{V_{\mathrm{reg}}} \longrightarrow N_{V_{\mathrm{reg}}} \longrightarrow 0. \tag{5.1.1}$$

In view of this, we call $N|_V$ the normal bundle of V.

Definition 5.1.1. (cf. [60]) The *virtual tangent bundle* τ_V of V is defined by $\tau_V = (TM - N)|_V$, regarded as an element in the complex K-theory $KU(V)$.

It is known that the equivalence class of this virtual bundle does not depend on the choice of the embedding of V in M.

Recalling the fact that the total Chern class is invertible in the cohomology ring, we define the total Chern class of the virtual tangent bundle by

$$c^*(\tau_V) = i^*(c^*(TM) \cdot c^*(N)^{-1}) \in H^*(V),$$

where $i : V \hookrightarrow M$ denotes the embedding. The i-th Chern class of τ_V is by definition the component of $c^*(\tau_V)$ in dimension $2i$, for $i = 1, \ldots, n$.

It is clear that if V is nonsingular, then its virtual tangent bundle is equivalent in $KU(V)$ to its usual tangent bundle and therefore the Chern classes of the virtual tangent bundle are the usual Chern classes.

5.2 Chern–Weil Theory for Virtual Bundles

If we have a complex vector bundle E_i on a C^∞ manifold M, for each $i = 0, \ldots, q$, we may consider the "virtual bundle" $\xi = \sum_{i=0}^{q}(-1)^i E_i$ as an element in the K-group $K(M)$ of M. We define its total Chern class $c(\xi)$ in $H^*(M)$ by

$$c(\xi) = \prod_{i=0}^{q} c(E_i)^{\varepsilon(i)},$$

where $\varepsilon(i) = (-1)^i$. The component of $c(\xi)$ in $H^{2j}(M)$ is the j-th Chern class $c^j(\xi)$ of ξ. In general, if φ is a symmetric polynomial, we may write $\varphi(\xi)$ as a polynomial in the Chern classes of ξ and express as a finite sum

$$\varphi(\xi) = \sum_k \varphi_k^{(0)}(E_0) \cdots \varphi_k^{(q)}(E_q),$$

where, for each i and k, $\varphi_k^{(i)}(E_i)$ is a polynomial in the Chern classes of E_i.

Letting $\nabla^{(i)}$ be a connection for E_i, $i = 0, \ldots, q$, we denote by $\nabla^\bullet$ the family of connections $(\nabla^{(q)}, \ldots, \nabla^{(0)})$. Then $\varphi(\xi)$ is the cohomology class of the differential form

$$\varphi(\nabla^\bullet) = \sum_k \varphi_k^{(0)}(\nabla^{(0)}) \wedge \cdots \wedge \varphi_k^{(q)}(\nabla^{(q)}).$$

In particular, $c(\xi)$ is the class of

$$c(\nabla^\bullet) = \prod_{i=0}^{q} c(\nabla^{(i)})^{\varepsilon(i)}$$

and $c^j(\xi)$ is the class of the homogeneous component $c^j(\nabla^\bullet)$ of degree $2j$ in $c(\nabla^\bullet)$.

Now suppose we have two families of connections $\nabla_\nu^\bullet = (\nabla_\nu^{(q)}, \ldots, \nabla_\nu^{(0)})$, $\nu = 0, 1$. Then, for a symmetric polynomial φ, we have a form $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$ satisfying

$$\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = -\varphi(\nabla_1^\bullet, \nabla_0^\bullet) \qquad \text{and} \qquad d\,\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = \varphi(\nabla_1^\bullet) - \varphi(\nabla_0^\bullet). \tag{5.2.1}$$

In fact, this is done as in the case of single vector bundles (cf. (1.4.2)). Thus, for each $i = 0, \ldots, q$, we consider the vector bundle $E_i \times \mathbb{R} \to M \times \mathbb{R}$ and let $\tilde{\nabla}^{(i)}$ be the connection for it given by $\tilde{\nabla}^{(i)} = (1-t)\nabla_0^{(i)} + t\nabla_1^{(i)}$. We set $\tilde{\nabla}^\bullet = (\tilde{\nabla}^{(q)}, \ldots, \tilde{\nabla}^{(0)})$ and define $\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = \pi_*(\varphi(\tilde{\nabla}^\bullet))$, where π is the projection $M \times [0,1] \to M$.

A similar construction works for an arbitrary collection of finite number of families of connections.

The form $\varphi(\nabla^\bullet)$ is closed and defines the class $\varphi(\xi)$. From the above, we see that the class $\varphi(\xi)$ depends only on ξ and not on the choice of $\nabla^\bullet$.

Now let

$$0 \longrightarrow E_q \xrightarrow{\psi_q} \cdots \longrightarrow E_1 \xrightarrow{\psi_1} E_0 \longrightarrow 0 \tag{5.2.2}$$

be a sequence of vector bundles on M, and, for each i, let $\nabla^{(i)}$ be a connection for E_i. We say that the family $(\nabla^{(q)}, \ldots, \nabla^{(0)})$ is compatible with the sequence if, for each i, the following diagram is commutative :

$$\begin{array}{ccc} A^0(M,E_i) & \xrightarrow{\nabla^{(i)}} & A^1(M,E_i) \\ {\scriptstyle \psi_i}\big\downarrow & & \big\downarrow{\scriptstyle 1\otimes\psi_i} \\ A^0(M,E_{i+1}) & \xrightarrow{\nabla^{(i+1)}} & A^1(M,E_{i+1}). \end{array}$$

If the above sequence is exact, there is always a family $\nabla^\bullet$=$(\nabla^{(q)},\ldots,\nabla^{(0)})$ of connections compatible with the sequence and for such a family we have ([14, (4.22) Lemma])

$$c^*(\nabla^\bullet) = 1 \quad \text{in particular} \quad c^*(\xi) = 1, \tag{5.2.3}$$

where $\xi = \sum_{i=0}^{q}(-1)^i E_i$. From this we have the following:

Proposition 5.2.1. *Suppose the sequence (5.2.2) is exact. Let* φ *be a symmetric polynomial and* $\nabla^\bullet = (\nabla^{(q)},\ldots,\nabla^{(0)})$*, a family of connections compatible with (5.2.2). Then*

$$\varphi(\check{\nabla}_0^\bullet) = \varphi(\nabla_0^{(0)}) \quad \textit{in particular} \quad \varphi(\check{\xi}) = \varphi(E_0),$$

where $\check{\nabla}^\bullet$ *denotes the family of connections* $(\nabla^{(q)},\ldots,\nabla^{(1)})$ *for the virtual bundle* $\check{\xi} = \sum_{i=1}^{q}(-1)^{i-1}E_i$*. Similarly for the other "partitions" of the virtual bundle* ξ*.*

Note that a similar statements hold for the Bott difference form of families of connections.

Let M be a C^∞ manifold, $\xi = \sum_{i=0}^{q}(-1)^i E_i$ a virtual bundle over M and φ a symmetric polynomial, as before. Also let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of M. Choosing a family of connections $\nabla_\nu^\bullet = (\nabla_\nu^{(q)},\ldots,\nabla_\nu^{(1)})$ for ξ on U_ν, $\nu = 0,1$, we have a cochain

$$\varphi(\nabla_\star^\bullet) = (\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)) \tag{5.2.4}$$

in $A^*(\mathcal{U})$. By (5.2.1), it is a cocycle and defines a class in the Čech-de Rham cohomology $H_D^*(\mathcal{U})$, which corresponds to the class $\varphi(\xi)$ via the isomorphism of Theorem 1.5.1.

Moreover, if we may choose $\nabla_0^\bullet$ so that $\varphi(\nabla_0^\bullet) = 0$, the cocycle $\varphi(\nabla_\star^\bullet)$ defines a class in the relative cohomology $H_D^*(\mathcal{U}, U_0)$. This idea is used in the localization theory of characteristic classes of virtual bundles.

5.3 Characteristic Numbers on Singular Varieties

Let V be an analytic variety of pure dimension n in a complex manifold M of dimension $n+k$. We set $V_{\text{reg}} = V \setminus \text{Sing}(V)$ as before. First, suppose V is compact and let $\widehat{U}$ be a neighborhood of V in M. Also, let $\widehat{\mathcal{U}} = \{\widehat{U}_0, \widehat{U}_1\}$ be an open covering of $\widehat{U}$ and $\{\widehat{R}_0, \widehat{R}_1\}$ a system of honey-comb cells adapted to $\widehat{\mathcal{U}}$ (cf. Sect. 1.5) such that V is transverse to $\widehat{R}_{01} = \partial\widehat{R}_0 = -\partial\widehat{R}_1$. We set $R_i = \widehat{R}_i \cap V$, $i = 0, 1$ and $R_{01} = \widehat{R}_{01} \cap V$. Then we may define the integration

$$\int_V : A^{2n}(\widehat{\mathcal{U}}) \to \mathbb{C} \qquad \text{by} \quad \int_V \sigma = \int_{R_0} \sigma_0 + \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01} \tag{5.3.1}$$

for $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ in $A^{2n}(\widehat{\mathcal{U}})$. It induces the integration on the cohomology

$$\int_V : H_D^{2n}(\widehat{\mathcal{U}}) \to \mathbb{C}. \tag{5.3.2}$$

We have $H_D^{2n}(\widehat{\mathcal{U}}) \simeq H^{2n}(\widehat{U}, \mathbb{C})$ and the above integration is compatible with the integration $\int_V : H^{2n}(\widehat{U}, \mathbb{C}) \to \mathbb{C}$ induced from the integration of $2n$-forms on $\widehat{U}$ over the $2n$-cycle V.

Now suppose V may not be compact. Let S be a compact set in V such that $V \setminus S \subset V_{\mathrm{reg}}$. Letting $\widehat{U}_1$ be a neighborhood of S in M and $\widehat{U}_0$ a tubular neighborhood of $V \setminus S$ in M with a C^∞ retraction $\rho : \widehat{U}_0 \to V \setminus S$, we consider the covering $\widehat{\mathcal{U}} = \{\widehat{U}_0, \widehat{U}_1\}$ of $\widehat{U} = \widehat{U}_0 \cup \widehat{U}_1$, which is an open neighborhood of V in M. As in Sect. 1.5, we set $A^r(\widehat{\mathcal{U}}, \widehat{U}_0) = \{\, \sigma \in A^r(\widehat{\mathcal{U}}) \mid \sigma_0 = 0 \,\}$.

Let $\widehat{R}_1$ be a compact real $2(n+k)$-dimensional manifold with C^∞ boundary in $\widehat{U}_1$ such that S is in its interior and that $\partial\widehat{R}_1$ is transverse to V. We set $R_1 = \widehat{R}_1 \cap V$, $R_{01} = -\partial R_1 = -\partial\widehat{R}_1 \cap V$. Then we can define the integration

$$\int_V : A^{2n}(\widehat{\mathcal{U}}, \widehat{U}_0) \to \mathbb{C} \qquad \text{by} \quad \int_V \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01} \tag{5.3.3}$$

for $\sigma = (0, \sigma_1, \sigma_{01})$ in $A^{2n}(\widehat{\mathcal{U}}, \widehat{U}_0)$. This again induces the integration on the cohomology

$$\int_V : H_D^{2n}(\widehat{\mathcal{U}}, \widehat{U}_0) \to \mathbb{C}. \tag{5.3.4}$$

Now suppose V is compact again and let S be as above. Then the following diagram is commutative :

$$\begin{array}{ccc} H_D^{2n}(\widehat{\mathcal{U}}, \widehat{U}_0) & \xrightarrow{\ j^*\ } & H_D^{2n}(\widehat{\mathcal{U}}) \\ \Big\downarrow {\scriptstyle \int_V} & & \Big\downarrow {\scriptstyle \int_V} \\ \mathbb{C} & \xrightarrow{\ =\ } & \mathbb{C}, \end{array} \tag{5.3.5}$$

where j^* denotes the canonical homomorphism.

Remark 5.3.1. In the above, the assumption that $V \setminus S$ is in the regular part V_{reg} is not necessary. However, with this condition, to define a cochain

$\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ in $A^{2n}(\widehat{\mathcal{U}})$ we only need to define σ_0 on $V \setminus S$, since there is a C^∞ retraction $\rho : \widehat{U}_0 \to V \setminus S$.

Again, let V be a variety of dimension n in a complex manifold M and S a compact set in V (V may not be compact) such that $V \setminus S \subset V_{\mathrm{reg}}$. Let $\widehat{U}_1$, $\widehat{U}_0$, $\widehat{\mathcal{U}} = \{\widehat{U}_0, \widehat{U}_1\}$ and $\widehat{U} = \widehat{U}_0 \cup \widehat{U}_1$ be as above. For a complex vector bundle E over $\widehat{U}$ and a homogeneous symmetric polynomial φ of degree n, we try to compute the restriction to V of the characteristic class $\varphi(E)$. The characteristic class $\varphi(E)$ in $H^{2n}_D(\widehat{\mathcal{U}}) \simeq H^{2n}(\widehat{U}, \mathbb{C})$ is represented by the cocycle $\varphi(\nabla_*)$ in $A^{2n}(\widehat{\mathcal{U}})$ given by

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on $\widehat{U}_0$ and $\widehat{U}_1$, respectively. Note that it is sufficient if ∇_0 is defined only on $U_0 = V \setminus S$ (see Remark 5.3.1). Suppose that there is some "geometric object" γ on V away from S, to which is associated a class $\mathcal{C}$ of connections for E on U_0 and that ∇_0 is "special," *i.e.*, ∇_0 belongs to $\mathcal{C}$, and φ is adapted to $\mathcal{C}$ (cf. Sect. 1.6.2). Then we have the vanishing

$$\varphi(\nabla_0) \equiv 0 \tag{5.3.6}$$

and the above cocycle $\varphi(\nabla_*)$ is in $A^{2n}(\widehat{\mathcal{U}}, \widehat{U}_0)$. Hence it defines a class $\varphi_S(E, \gamma)$ in the relative cohomology $H^{2n}_D(\widehat{\mathcal{U}}, \widehat{U}_0)$, which is sent to the class $\varphi(E)$ by the canonical homomorphism $j^* : H^{2n}_D(\widehat{\mathcal{U}}, \widehat{U}_0) \to H^{2n}_D(\widehat{\mathcal{U}})$. The class $\varphi_S(E, \gamma)$ does not depend on the choice of the special connection ∇_0 or the connection ∇_1.

Now suppose that S has a finite number of connected components $(S_\lambda)_\lambda$. For each λ, we take a neighborhood $\widehat{U}_\lambda$ of S_λ in U_1 so that the $\widehat{U}_\lambda$'s are disjoint one another. Let $\widehat{R}_\lambda$ be a compact $2(n+k)$-dimensional manifold with C^∞ boundary in $\widehat{U}_\lambda$ such that S_λ is in its interior and that $\partial \widehat{R}_\lambda$ is transverse to V. We set $R_\lambda = \widehat{R}_\lambda \cap V$ and $R_{0\lambda} = -\partial \widehat{R}_\lambda$.

Definition 5.3.1. We define the residue $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ by

$$\mathrm{Res}_\varphi(\gamma, E; S_\lambda) = \int_{R_\lambda} \varphi(\nabla_1) + \int_{R_{0\lambda}} \varphi(\nabla_0, \nabla_1).$$

From the above considerations and the commutative diagram (5.3.5), we have the following residue theorem on singular varieties:

Theorem 5.3.7. *In the above situation,*

(1) *For each connected component S_λ of S, we have the residue* $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$, *which is determined by the local behavior of γ near S_λ and is given by the formula in Definition 5.3.1.*

(2) *If V is compact,*

$$\sum_{\lambda} \mathrm{Res}_{\varphi}(\gamma, E; S_{\lambda}) = \int_V \varphi(E).$$

Remark 5.3.2. 1. Similar considerations work for virtual bundles as long as we have some vanishing as (5.3.6).

2. For the case of polynomials of degree lower than n, where the residues are defined in the homology of S, see [156, Ch.VI, 4].

5.4 The Virtual Index

Let V be a local complete intersection of dimension n defined by a section of a holomorphic vector bundle N of rank k over a complex manifold M of dimension $n+k$ (cf. Sect. 5.1). Let $\tau_V = (TM - N)|_V$ be the virtual tangent bundle of V, as before. In the sequel, we set $\tau = TM - N$.

Let S be a compact set in V containing the singular set $\mathrm{Sing}(V)$ of V. Suppose we have a C^∞ vector field v on $V \setminus S$. Then we will see that the top Chern class $c^n(\tau_V)$ of the virtual tangent bundle is localized on S, the virtual index being defined to be the associated residue.

Let $\widehat{U}_1$, $\widehat{U}_0$, $\widehat{\mathcal{U}} = \{\widehat{U}_0, \widehat{U}_1\}$ and $\widehat{U} = \widehat{U}_0 \cup \widehat{U}_1$ be as in Sect. 5.3. Also, let ∇_0 and ∇'_0 be connections for $TM|_{V_{\mathrm{reg}}}$ and $N|_{V_{\mathrm{reg}}}$, respectively, on $U_0 = V \setminus S$ and let ∇_1 and ∇'_1 be connections for TM and N, respectively, on $\widehat{U}_1$. We set $\nabla^\bullet_0 = (\nabla_0, \nabla'_0)$ and $\nabla^\bullet_1 = (\nabla_1, \nabla'_1)$. Then the characteristic class $c^n(\tau)$ in $H^{2n}_D(\widehat{\mathcal{U}}) \simeq H^{2n}(\widehat{U}, \mathbb{C})$ is represented by the cocycle $c^n(\nabla^\bullet_*)$ in $A^{2n}(\widehat{\mathcal{U}})$ given by

$$c^n(\nabla^\bullet_*) = (c^n(\nabla^\bullet_0), c^n(\nabla^\bullet_1), c^n(\nabla^\bullet_0, \nabla^\bullet_1)).$$

Now we take connections ∇''_0, ∇_0 and ∇'_0 for TV_{reg}, $TM|_{V_{\mathrm{reg}}}$ and $N_{V_{\mathrm{reg}}}$, respectively, on U_0 so that

(1) ∇''_0 is v-trivial, *i.e.*, $\nabla''_0(v) = 0$, and that
(2) the triple $(\nabla''_0, \nabla_0, \nabla'_0)$ is compatible with (5.1.1):

$$0 \longrightarrow TV_{\mathrm{reg}} \longrightarrow TM|_{V_{\mathrm{reg}}} \longrightarrow N_{V_{\mathrm{reg}}} \longrightarrow 0.$$

Then we have

$$c^n(\nabla^\bullet_0) = c^n(\nabla''_0) = 0$$

because of (1) and (2) above (cf. Propositions 5.2.1 and 1.6.1) and the above cocycle $c^n(\nabla^\bullet_*)$ is in $A^{2n}(\widehat{\mathcal{U}}, \widehat{U}_0)$. Hence it defines a class $c^n(\tau, v)$ in $H^{2n}_D(\widehat{\mathcal{U}}, \widehat{U}_0)$, which is sent to $c^n(\tau)$ by the canonical homomorphism $j^* : H^{2n}_D(\widehat{\mathcal{U}}, \widehat{U}_0) \to H^{2n}_D(\widehat{\mathcal{U}})$. We define the virtual index $\mathrm{Ind}_{\mathrm{Vir}}(v, S)$ to be the corresponding residue. If S admits a finite number of connected components

$(S_\lambda)_\lambda$, then we have the virtual index $\mathrm{Ind}_{\mathrm{Vir}}(v, S_\lambda)$ for each λ. Let R_λ and $R_{0\lambda}$ be as in Sect. 5.3. Then we may rephrase the definition of the virtual index as (cf. [111, 149]).

Definition 5.4.1. The *virtual index* of v at S_λ is defined by

$$\mathrm{Ind}_{\mathrm{Vir}}(v, S_\lambda) = \int_{R_\lambda} c^n(\nabla_1^\bullet) + \int_{R_{0\lambda}} c^n(\nabla_0^\bullet, \nabla_1^\bullet).$$

It is not difficult to show the following (cf. [156, Ch.IV, Lemma 3.3])

Lemma 5.4.1. *If S_λ is in V_{reg}, $\mathrm{Ind}_{\mathrm{Vir}}(v, S_\lambda) = \mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda)$.*

From Theorem 5.3.7, we have the following :

Theorem 5.4.1. *In the above situation, if V is compact,*

$$\sum_\lambda \mathrm{Ind}_{\mathrm{Vir}}(v, S_\lambda) = \int_V c^n(\tau).$$

5.5 Identification with GSV Index When Singularities are Isolated

The following proposition is proved as [111, Lemma 5]. Here we reproduce the proof as given in [149].

Theorem 5.5.1. *Let x be an isolated singular point of a variety V as above and let $\widehat{v}$ be a C^∞ vector field on a neighborhood $\widehat{U}$ of x in M, which is possibly singular only at x and is tangent to V_{reg}. Then, for the vector field v on $U \setminus \{z\}$, $U = \widehat{U} \cap V$, induced by $\widehat{v}$,*

$$\mathrm{Ind}_{\mathrm{Vir}}(v, x) = \mathrm{Ind}_{\mathrm{GSV}}(v, x).$$

Proof. By taking a smaller $\widehat{U}$, if necessary, we may assume that there is a system $f = (f_1, \dots, f_k)$ of holomorphic functions on $\widehat{U}$ which generates the ideal of functions vanishing on $V \cap \widehat{U}$. Denoting by $C(f)$ the critical set of $f : \widehat{U} \to \mathbb{C}^k$, we set $\widehat{U} = \widehat{U} \setminus C(f)$. Then we have the exact sequence, which extends (5.1.1) (restricted to $V \cap \widehat{U}$) :

$$0 \longrightarrow Tf|_{\widehat{U}} \longrightarrow TM|_{\widehat{U}} \longrightarrow N|_{\widehat{U}} \longrightarrow 0, \tag{5.5.2}$$

where $Tf|_{\widehat{U}}$ is the bundle of vectors tangent to the fibers of f. Starting from $\widehat{v}$, we may construct a C^∞ vector field v' on $\widehat{U}$ so that it is tangent to (the regular part of) each fiber of f, its singular set $\widehat{S}$ contains $C(f)$ and that the intersection of $\widehat{S}$ and the fiber is compact. This is done by taking a C^∞

splitting of (5.5.2). Let v_t denote the restriction of v' to the fiber $f^{-1}(t)$, for t in $\mathbb{C}^k$ near 0, and S_t the intersection $\widehat{S} \cap f^{-1}(t)$, which contains the singular set of $f^{-1}(t)$. We compute $\mathrm{Ind}_{\mathrm{Vir}}(v_t, S_t)$, the sum of virtual indices of v_t over the components of S_t, using (the restriction to V_t of) connections as follows. We take connections ∇_0'', ∇_0 and ∇_0' for $Tf|_{\widehat{U}}$, $TM|_{\widehat{U}}$ and $N_{\widehat{U}}$, respectively, on $\widehat{U} \setminus \widehat{S}$ so that

(1) ∇_0'' is v'-trivial and that
(2) the triple $(\nabla_0'', \nabla_0, \nabla_0')$ is compatible with (5.5.2).

We let ∇_1 and ∇_1' be arbitrary connections for TM and N, respectively, on $\widehat{U}$. Then we see that $\mathrm{Ind}_{\mathrm{Vir}}(v_t, S_t)$ depends continuously on t. For a regular value t, this is $\mathrm{Ind}_{\mathrm{GSV}}(v, a)$, which is an integer (Lemma 5.4.1 and Theorem 3.2.1). Thus it does not depend on t, since the regular values are dense. While for $t = 0$, this is equal to $\mathrm{Ind}_{\mathrm{Vir}}(v, a)$ and the theorem follows.

5.6 A Generalization of the Adjunction Formula

The classical Adjunction Formula says that, if C is a compact nonsingular curve (Riemann surface) in a complex surface M (complex manifold of dimension 2), then we have

$$2 - 2g = -(K_M + C) \cdot C$$

where K_M is the canonical divisor of M, g is the genus of C and the dot means intersection of cycles. This formula follows from classical relations among characteristic classes. In fact, $2-2g$ is $\chi(C)$ and this equals $c^1(TC)[C]$, the self intersection number C^2 is the Poincaré dual of the first Chern class of the normal bundle of C, *i.e.*, $C^2 = c^1(N_C)[C]$. The canonical divisor K_M is the Poincaré dual of $-c^1(TM)$, essentially by definition; hence the intersection product $K_M \cdot C$ equals $-c^1(TM|_C)[C]$. Therefore the Adjunction formula follows from the exact sequence

$$0 \longrightarrow TC \longrightarrow TM|_C \longrightarrow N_C \longrightarrow 0.$$

This formula was generalized by Kodaira [99, 2.2] for a possibly singular curve C in a complex surface M as:

$$\chi(\widetilde{C}) = -(K_M + C) \cdot C + \sum_{i=1}^{r} c(C, x_i), \tag{5.6.1}$$

where $\widetilde{C}$ is a nonsingular model of C and $c(C, x_i)$ is an invariant of C at the singular point x_i, which is related to the Milnor number $\mu(C, x_i)$ by $c(C, x_i) = \mu(C, x_i) + s_i - 1$ with s_i the number of (local) branches of C at

x_i. Since $\chi(\widetilde{C}) - \sum_{i=1}^r (s_i - 1) = \chi(C)$, (5.6.1) is equivalent to

$$\chi(C) = -(K_M + C) \cdot C + \sum_{i=1}^r \mu(C, x_i), \tag{5.6.2}$$

From the results of the previous sections, we have a generalization of this formula to the higher dimensional case:

Theorem 5.6.3. *Let V be a local complete intersection of dimension n in M defined by a section. If V is compact and has only isolated singularities $x_1, \ldots, x_r$, whose Milnor numbers are denoted by μ_i, then*

$$\chi(V) = \int_V c^n(\tau) + (-1)^{n+1} \sum_{i=1}^r \mu_i.$$

Proof. First we claim that if V is as above, then there exists a C^∞ vector field v on V, singular at the x_i's and at a (possibly empty) finite set of other points. In fact, from [121] we know that there is a C^∞ vector field $\widehat{v}_i$ on a neighborhood B_i of each x_i in M, which is singular only at x_i and is tangent to V. Let $D_i = B_i \cap V$ and let v_i be the restriction of $\widehat{v}_i$ to D_i. Then $V^* = V \setminus \cup_{i=1}^r \mathrm{Int}(D_i)$ is a smooth manifold with boundary, and the v_i's determine a nonsingular vector field on the boundary of V^*. This can be extended to a C^∞ vector field on all of V^*, with at most a finite number of singularities, proving the claim. Now, let v be a vector field on V singular at the x_i's and at a (possibly empty) finite set of other points $x_{r+1}, \ldots, x_{r+s}$. By Theorem 3.2.2 one has:

$$\sum_{i=1}^{r+s} \mathrm{Ind}_{\mathrm{GSV}}(v, x_i) = \chi(V) + (-1)^n \sum_{i=1}^r \mu_i.$$

Since the singularities of V are all isolated, Theorem 5.5.1 above says that in the above formula, all GSV indices can be considered to be virtual indices. Finally Theorem 5.4.1 tells us that the total sum of virtual indices equals $\int_V c^n(\tau)$.

Note that we need only the compactness of V but not of M.

Remark 5.6.1. 1. If V is nonsingular, we have $c^n(\tau_V) = c^n(TV)$. Hence the formula in Theorem 5.6.3 reduces to the "Gauss-Bonnet formula"

$$\int_V c^n(V) = \chi(V).$$

2. If V is a complete intersection in $M = \mathbb{CP}^{n+k}$, N is determined by its multi-degree $(d_1, \ldots, d_k)$ and we have (cf. e.g., [84, §22])

$$\int_V c^n(\tau) = \left[(1+h)^{n+k+1} \cdot \prod_{i=1}^{k} \frac{d_i}{1+d_i\,h}\right]_n , \tag{5.6.4}$$

where h denotes the first Chern class of the hyperplane bundle and $[\ \ ]_n$ the coefficient of h^n in $[\ \ ]$. Thus we may compute $\chi(V)$ from Theorem 5.6.3.

Let V_0 be a nonsingular complete intersection in $\mathbb{CP}^{n+k}$ of dimension n with the same multi-degree as V. Then we have $\chi(V_0) = \int_{V_0} c^n(V_0)$, which is also given by the right hand side of (5.6.4). Hence we have the following formula, which is readily proved by a direct argument as well (cf. [45, Ch.5, (4.4) Corollary]) :

$$\chi(V) = \chi(V_0) + (-1)^{n+1} \sum_{i=1}^{s} \mu(V, p_i).$$

3. A generalized Milnor number is defined for each compact connected component of the singular set of a hypersurface and a formula for the sum of these numbers is proved in [127]. The formula coincides with the one in Theorem 5.6.3, if the singularities are isolated. See also [130].

5.7 An Integral Formula for the Virtual Index

In this section, we give an integral formula for the virtual index of a holomorphic vector field at an isolated singular point. This is done in [111] and is a special case of more general residue theory for holomorphic vector fields.

Since the problem is local, let $\widehat{U}$ be a neighborhood of the origin 0 in $\mathbb{C}^{n+k}$ and V a complete intersection in $\widehat{U}$ defined by $h_1 = \cdots = h_k = 0$, with isolated singularity at 0. Let $\widehat{v}$ be a holomorphic vector field on $\widehat{U}$, which is tangent to and nonsingular on $V \setminus \{0\}$. We may choose a coordinate system $(z_1, \ldots, z_{n+k})$ on $\widehat{U}$ (see Theorem 6.3.1 below) so that, if we write as $\widehat{v} = \sum_{i=1}^{n+k} a_i \frac{\partial}{\partial z_i}$, the set of common zeros of $a_1, \ldots, a_n, h_1, \ldots, h_k$ consists of only 0. Let A denote the $(n+k) \times (n+k)$ matrix whose (i,j) entry is $\frac{\partial a_i}{\partial z_j}$ and C the $k \times k$ matrix whose (i,j) entry c_{ij} is determined by the "tangency condition" $\widehat{v}(h_i) = \sum_{i=1}^{k} c_{ij} h_j$. Let $\lambda_n = [c(A) \cdot c(C)^{-1}]_n$ denote the coefficient of t^n in the power series expansion of

$$\det(I + tA) \cdot \det(I + tC)^{-1}.$$

With these, we have the following theorem ([111]):

Theorem 5.7.1. *The virtual index at* 0 *of the restriction* v *of* $\widehat{v}$ *to* V *is given by*

$$\mathrm{Ind}_{\mathrm{Vir}}(v,0)=\frac{1}{(2\pi\sqrt{-1})^n}\int_{\Gamma}\frac{\lambda_n\,dz_1\wedge\cdots\wedge dz_n}{a_1\cdots a_n},$$

where Γ is the n-cycle in V given by

$$\Gamma=\{\,q\in\widehat{U}\cap V\mid |a_i(q)|=\varepsilon_i,\quad i=1,\dots,n\,\}$$

for small positive numbers ε_i. It is oriented so that $d\theta_1\wedge\cdots\wedge d\theta_n\geq 0$, $\theta_i=\arg a_i$.

This is a special case of a more general formula for the residues of virtual tangent bundle, see Theorem 6.3.11 below.

Example 5.7.1. Let $d_1,\dots,d_{n+1}$ be nonzero rational numbers and f a polynomial in $(z_1,\dots,z_{n+1})$. We say that f is a *weighted homogeneous polynomial of type* $(d_1,\dots,d_{n+1})$ if it is a linear combination of monomials $z_1^{p_1}\cdots z_{n+1}^{p_{n+1}}$ such that $\sum_{i=1}^{n+1}p_i/d_i=1$. This is equivalent to saying that there exist nonzero integers $q_1,\dots,q_{n+1}$ and a positive integer d such that

$$f(t^{q_1}z_1,\dots,t^{q_{n+1}}z_{n+1})=t^d\,f(z_1,\dots,z_{n+1}).$$

In this case, $d_i=d/q_i$, $i=1,\dots,n+1$.

Let V be a hypersurface in $\mathbb{C}^{n+1}=\{(z_1,\dots,z_{n+1})\}$ defined by a weighted homogeneous polynomial f of type $(d_1,\dots,d_{n+1})$ with isolated singularity at the origin 0. For the holomorphic vector field $\widehat{v}=\sum_{i=1}^{n+1}z_i/d_i\cdot\partial/\partial z_i$, we have $\widehat{v}(f)=f$ and thus V is invariant by $\widehat{v}$. Let v denote the vector field on the nonsingular part of V induced from $\widehat{v}$. We assume that f is regular in z_{n+1}. This implies that d_{n+1} is a positive integer and f is regular in z_{n+1} of order d_{n+1}. If we let $a_i=z_i/d_i$, $i=1,\dots,n+1$, we have $V(a_1,\dots,a_n,f)=\{0\}$. Using the formula in Theorem 5.7.1, we compute

$$\mathrm{Ind}_{\mathrm{Vir}}(v,0)=1+(-1)^n(d_1-1)(d_2-1)\cdots(d_{n+1}-1).$$

Note that, since v is transverse to the boundary of the Milnor fiber of f, $\mathrm{Ind}_{\mathrm{Vir}}(v,0)$ is also equal to the Euler number $1+(-1)^n\mu(V,0)$ of the Milnor fiber, where $\mu(V,0)$ denotes the Milnor number of V at 0. Thus we reprove the formula

$$\mu(V,0)=(d_1-1)(d_2-1)\cdots(d_{n+1}-1)$$

for the Milnor number ([122]).

Remark 5.7.1. Yayoi Nakamura (Kinki University, Osaka, Japan) developed a systematic computer program to calculate Grothendieck residues. She used it and the formula of Theorem 5.7.1 to compute the virtual index (*i.e.*, GSV index) of certain holomorphic vector fields, given as examples in [17]. She obtained the same numbers as the ones in [17] (exposed during the in Russian-Japanese Conference November 2007, Tokyo).

Chapter 6
The Case of Holomorphic Vector Fields

Abstract We have seen that for vector fields, there are indices such as the Poincaré–Hopf index and the virtual index, that arise from localizations of certain Chern classes. If the vector field is holomorphic, the localization theory becomes richer because of the Bott vanishing theorem, and this produces further interesting residues. This theory can be developed for general singular foliations on certain singular varieties. We consider here the case of holomorphic vector fields and the slightly more general case of one dimensional singular foliations. We refer to [156] for a systematical treatment of the general case.

Here we have three types of residues:
(1) Baum–Bott residues and generalizations to singular varieties,
(2) Camacho–Sad index and various generalizations,
(3) Variations and generalizations.

In all the above cases the residues arise from a Bott type vanishing theorem, which in turn comes from an action of the vector field or the foliation on some vector bundle or virtual bundle. The residues of type (1) were first introduced by R. Baum and P. Bott in [13,14]. In general these arise from the action of the foliation on the normal sheaf of the foliation. The Camacho–Sad index (2) was introduced in [42] and was effectively used to prove the existence of a separatrix at a singular point of a holomorphic vector field on the complex plane. Nowadays there are many generalizations of this index, see Remark 6.3.3 below. These residues arise from the action of the foliation on the normal bundle of an invariant subvariety. The residues of type (3) were first introduced by B. Khanedani and T. Suwa in [93] and generalized in [113]; see also the related articles [39] and [40] by M. Brunella. These type of residues arise from the action of the foliation on the ambient tangent bundle. These three types of residues are listed above in historical order, but they are explained below in the reversed order, for logical reasons.

In each case, the residue at an isolated singularity can be expressed in terms of a Grothendieck residue on singular variety.

J.-P. Brasselet et al., *Vector Fields on Singular Varieties*,
Lecture Notes in Mathematics 1987, DOI 10.1007/978-3-642-05205-7_6,

6.1 Baum–Bott Residues of Holomorphic Vector Fields

Let M be a complex manifold of dimension m. For a holomorphic vector field (holomorphic section of TM) v on M, we set

$$S(v) = \{ p \in M \mid v(p) = 0\}$$

and call it the singular set of v. It is an analytic variety in M. We set $M_0 = M \setminus S(v)$.

Definition 6.1.1. We say that a holomorphic vector bundle E on M_0 is a holomorphic v-*bundle*, if it admits a holomorphic action of v, *i.e.*, if there exists a $\mathbb{C}$-linear map

$$\alpha_v : A^0(M_0, E) \longrightarrow A^0(M_0, E)$$

satisfying the following conditions, for f in $A^0(M_0)$ and s in $A^0(M_0, E)$:
(1) $\alpha_v(fs) = v(f)s + f\alpha_v(s)$ and
(2) $\alpha_v(s)$ is holomorphic if s is.

Definition 6.1.2. We say that a connection ∇ for E is a v-*connection* if
(1) $\nabla s(v) = \alpha_v(s)$, for $s \in A^0(M_0, E)$, and
(2) ∇ is of type $(1, 0)$.

Here we recall that a connection ∇ for a holomorphic vector bundle E is said to be of type $(1, 0)$, if the entries of the connection matrix of ∇ with respect to a holomorphic frame of E are forms of type $(1, 0)$.

We quote the following theorem (cf. [14, 19, 156]):

Theorem 6.1.1. (Bott vanishing theorem I) *In the above situation, if* ∇ *is a* v*-connection for* E*, for a homogeneous symmetric polynomial* φ *of degree* m*, one has:*

$$\varphi(\nabla) \equiv 0.$$

Note that a similar result holds for the difference form of a finite number of connections and for families of v-connections of virtual bundles.

We see that TM_0 becomes a holomorphic v-bundle by the action

$$\alpha_v : A^0(M_0, TM_0) \longrightarrow A^0(M_0, TM_0)$$

given by $\alpha_v(w) = [v, w]$. Note that there exists a v-connection which is v-trivial.

We will see that, for a homogeneous symmetric polynomial φ of degree m, the class $\varphi(TM)$ is localized at $S(v)$. Here the vector field v is the "geometric object" of Sect. 1.6.2, a "special connection" is a v-connection and the relevant "vanishing theorem" is Theorem 6.1.1.

Set $S = S(v)$. Then from the arguments in Sect. 1.6.2, we have a class in $H^{2m}(M, M \setminus S; \mathbb{C})$, which we denote by $\varphi_S(TM, v)$ and call the localization of $\varphi(TM)$ by v at S. It is sent to $\varphi(TM)$ by the canonical homomorphism $j^* : H^{2m}(M, M \setminus S; \mathbb{C}) \to H^{2m}(M, \mathbb{C})$.

Definition 6.1.3. Suppose $S = S(v)$ is compact. The *Baum–Bott residue* $\mathrm{Res}_\varphi(v, TM; S)$ (sometimes abbreviated as $\mathrm{Res}_\varphi(v, S)$) is the image of the class $\varphi_S(TM, v)$ by the Alexander isomorphism

$$H^{2m}(M, M \setminus S; \mathbb{C}) \xrightarrow{\sim} H_0(S, \mathbb{C}).$$

The residue $\mathrm{Res}_\varphi(v, TM; S)$ is a complex number given by the right hand side of (1.6.4).

If S has only a finite number of connected components $(S_\lambda)_\lambda$, we have the residue $\mathrm{Res}_\varphi(v, TM; S_\lambda)$ for each λ.

The above residues are originally introduced in [13, 14], where they are defined using forms with compact support.

From the above argument and Theorem 1.6.5, we have the following theorem.

Theorem 6.1.2. *In the above situation,*

(1) *For each connected component S_λ of $S(v)$, we have a well-defined residue $\mathrm{Res}_\varphi(v, TM; S_\lambda)$.*

(2) *If M is compact,*

$$\sum_\lambda \mathrm{Res}_\varphi(v, TM; S_\lambda) = \int_M \varphi(TM).$$

Remark 6.1.1. We may choose the connection ∇_0 above so that it is also v-trivial. Thus, if $\varphi = c^m$, the residue $\mathrm{Res}_{c^m}(v, TM; S)$ coincides with the Poincaré–Hopf index; $\mathrm{Res}_{c^m}(v, TM; S) = \mathrm{Ind}_{\mathrm{PH}}(v, S)$ and the formula in the above theorem becomes the Poincaré–Hopf formula.

Now we compute the residue at an isolated singular point. In the definition of the residue, if S consists of a point, we may choose as U a coordinate neighborhood so that TM is trivial on U. Since the problem is local, we assume that U is a neighborhood of the origin 0 in $\mathbb{C}^m = \{(z_1, \dots, z_m)\}$ and that $S = \{0\}$. We write the vector field v explicitly as

$$v = \sum_{i=1}^m a_i \frac{\partial}{\partial z_i}$$

with a_i holomorphic functions on U and set $A = (\partial a_i / \partial z_j)$.

In general, for a square matrix H whose entries are holomorphic functions near 0 and for elementary symmetric functions σ_i of m variables, we define the holomorphic functions $\sigma_i(H)$ by

$$\det(I + tH) = 1 + t\sigma_1(H) + \cdots + t^m \sigma_m(H).$$

We set $\varphi(H) = P(\sigma_1(H), \ldots, \sigma_m(H))$ for a symmetric polynomial $\varphi = P(\sigma_1, \ldots, \sigma_m)$.

With these we have the following theorem, which is also originally due to [13, 14], where it is proved in the context of forms with compact support. See [156, Ch.III, Theorem 5.5] for a proof using the Čech-de Rham cohomology, the technique being originally due to [110].

Theorem 6.1.3. *In the above situation, for a homogeneous symmetric polynomial φ of degree m,*

$$\operatorname{Res}_\varphi(v, 0) = \operatorname{Res}_0 \begin{bmatrix} \varphi(A) dz_1 \wedge \cdots \wedge dz_m \\ a_1, \ldots, a_m \end{bmatrix}.$$

Remark 6.1.2. 1. The above definition of the matrix $\varphi(H)$ differs from that in [156] by a normalization constant, *i.e.*, in [156], it is defined as $\varphi(H) = P(c^1(H), \ldots, c^m(H))$, $c^i(H) = \left(\sqrt{-1}/2\pi\right)^i \sigma_i(H)$.

2. In particular, if $\varphi = c^m = \left(\sqrt{-1}/2\pi\right)^m \sigma_m$, noting that

$$\sigma_m(A)\, dz_1 \wedge \cdots \wedge dz_m = \det(A)\, dz_1 \wedge \cdots \wedge dz_m = da_1 \wedge \cdots \wedge da_m,$$

we have

$$\operatorname{Res}_{c^m}(v, 0) = \operatorname{Res}_0 \begin{bmatrix} da_1 \wedge \cdots \wedge da_m \\ a_1, \ldots, a_m \end{bmatrix},$$

which represents the Poincaré–Hopf index $\operatorname{Ind}_{\mathrm{PH}}(v, 0)$ of v at 0 (cf. Example 1.6.2).

Example 6.1.1. Let v be the vector field on $\mathbb{C}^m = \{(z_1, \ldots, z_m)\}$ given by

$$v = \sum_{i=1}^m \lambda_i z_i \frac{\partial}{\partial z_i}, \qquad \lambda_i \in \mathbb{C}.$$

Then, since A in this case is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_m$, the residue with respect to a polynomial φ of degree m is given by

$$\operatorname{Res}_\varphi(v, 0) = \frac{\varphi(\lambda_1, \ldots, \lambda_m)}{\lambda_1 \cdots \lambda_m}.$$

Thus, in general, the Baum–Bott residues are not necessarily integers (and even not real numbers).

6.2 One-Dimensional Singular Foliations

On complex manifolds, we encounter "one-dimensional foliations" more often than global holomorphic vector fields and the generalization of the previous theory to this case is not difficult.

Let M be a complex manifold of dimension m. First we give the following definition.

Definition 6.2.1. A dimension one singular holomorphic foliation $\mathcal{F}$ on M is determined by a system $\{(U_\alpha, v_\alpha, f_{\alpha\beta})\}$, where $\{U_\alpha\}$ is an open covering of M, v_α is a holomorphic vector field on U_α, for each α, and $f_{\alpha\beta}$ is a nonvanishing holomorphic function on $U_\alpha \cap U_\beta$, for each pair (α, β), such that

$$v_\beta = f_{\alpha\beta} v_\alpha \qquad \text{on} \qquad U_\alpha \cap U_\beta. \tag{6.2.1}$$

Let $\mathcal{F} = \{(U_\alpha, v_\alpha, f_{\alpha\beta})\}$ be a one-dimensional foliation. We call v_α a generator of $\mathcal{F}$ on U_α. If we denote by $S(v_\alpha)$ the set of zeros of v_α on U_α, since $S(v_\alpha)$ and $S(v_\beta)$ coincide in $U_\alpha \cap U_\beta$ by (6.2.1), the union $\bigcup_\alpha S(v_\alpha)$ is an analytic set in M, which we call the singular set of the foliation $\mathcal{F}$ and denote by $S(\mathcal{F})$. On $M \setminus S(\mathcal{F})$, $\mathcal{F}$ defines a nonsingular foliation of dimension one. The integral curves of v_α in $U_\alpha \setminus S(v_\alpha)$ patch together to obtain a decomposition of $M \setminus S(\mathcal{F})$ into a disjoint union of one-dimensional submanifolds (leaves of the foliation). Since the system $\{f_{\alpha\beta}\}$ satisfies the cocycle condition, it determines a line bundle, which we denote by F and call the *tangent bundle of the foliation*. We have a vector bundle homomorphism

$$\iota : F \longrightarrow TM,$$

which assigns to a section s of F represented by a collection (f_α) of holomorphic functions f_α the vector field $v = f_\alpha v_\alpha$, which is independent of α by (6.2.1). The homomorphism ι fails to be injective exactly on $S(\mathcal{F})$. Hence on $M_0 = M \setminus S(\mathcal{F})$, we have the quotient bundle $N_{F_0} = TM_0/F_0$ (the normal bundle of the foliation), where $F_0 = F|_{M_0}$, so that we have the following exact sequence on M_0 :

$$0 \longrightarrow F_0 \stackrel{\iota}{\longrightarrow} TM_0 \stackrel{\eta}{\longrightarrow} N_{F_0} \longrightarrow 0. \tag{6.2.2}$$

We call $TM - F$ the virtual normal bundle of the foliation.

In the case of one-dimensional foliations, the relevant vanishing theorem comes from an "action" of F_0.

Definition 6.2.2. Let F_0 be as above. An *action* of F_0 on a holomorphic vector bundle E over M_0 is a $\mathbb{C}$-bilinear map

$$\alpha : A^0(M_0, F) \times A^0(M_0, E) \longrightarrow A^0(M_0, E)$$

satisfying the following conditions, for f in $A^0(M_0)$, u and v in $A^0(M_0,F)$ and s in $A^0(M_0,E)$:
(1) $\alpha([u,v],s)=\alpha(u,\alpha(v,s))-\alpha(v,\alpha(u,s))$,
(2) $\alpha(fu,s)=f\alpha(u,s)$,
(3) $\alpha(u,fs)=u(f)s+f\alpha(u,s)$ and
(4) $\alpha(u,s)$ is holomorphic whenever u and s are.

A vector bundle E with an action of F_0 is called an F_0*-bundle.*

Definition 6.2.3. Let α be an action of F_0 on E. An F_0*-connection* (or α*-connection*, if it is necessary to specify the action) for E is a connection for E which satisfies the following conditions:
(1) $\nabla s(u)=\alpha(u,s)$, for $s\in A^0(M_0,E)$ and $u\in A^0(M_0,F)$, and
(2) ∇ is of type $(1,0)$.

Remark 6.2.1. Let v be a nonvanishing holomorphic vector field on M_0 and F_v the subbundle of TM_0 spanned by v. Let E be a holomorphic vector bundle and suppose we have an action of v on E (cf. Definition 6.1.1). Then E becomes an F_v-bundle with the action α given by

$$\alpha(fv,s)=f\alpha_v(s).$$

(Note that every element u of $A^0(M_0,F)$ is of the form $u=fv$ for some $f\in A^0(M_0)$.) In this case a v-connection is an F_v-connection.

We also have the following

Theorem 6.2.3. (Bott vanishing theorem II) *In the above situation, if* ∇ *is a* F_0*-connection for* E*, then, for a homogeneous symmetric polynomial* φ *of degree* m*, one has:*

$$\varphi(\nabla)\equiv 0.$$

Now we see that the map

$$\alpha : A^0(M_0,F_0)\times A^0(M_0,N_{F_0})\longrightarrow A^0(M_0,N_{F_0})$$

defined by $\alpha(v,\eta(w))=\eta([v,w])$ is well-defined and makes N_{F_0} an F_0-bundle.

Let $\nu_{\mathcal{F}}=TM-F$ be the virtual normal bundle of the foliation $\mathcal{F}$. We will see that, for a homogeneous symmetric polynomial φ of degree m, the characteristic class $\varphi(\nu_{\mathcal{F}})$ is localized at $S(\mathcal{F})$.

Let $S=S(\mathcal{F})$, $U_0=M\setminus S$ and U_1 an open neighborhood of S in M. Let $\nabla_0^\bullet=(\nabla_0',\nabla_0)$ be a pair of connections for F and TM, respectively, on U_0 and $\nabla_1^\bullet=(\nabla_1',\nabla_1)$ one for F and TM on U_1. Then $\varphi(\nu_{\mathcal{F}})$ is represented by the cocycle

$$\varphi(\nabla_*^\bullet)=(\varphi(\nabla_0^\bullet),\varphi(\nabla_1^\bullet),\varphi(\nabla_0^\bullet,\nabla_1^\bullet))$$

in $A^{2m}(\widehat{\mathcal{U}})$. Now let ∇ be an F_0-connection for N_{F_0} on U_0 and choose (∇_0',∇_0) so that the triple $(\nabla_0',\nabla_0,\nabla)$ is compatible with the sequence (6.2.2) on U_0. Then, by Proposition 5.2.1 and Theorem 6.2.3,

$$\varphi(\nabla_0^\bullet) = \varphi(\nabla) = 0$$

and thus $\varphi(\nabla_*^\bullet)$ is in $A^{2m}(\widehat{\mathcal{U}}, U_0)$ and it defines a class in $H^{2m}(M, M \setminus S; \mathbb{C})$, which we denote by $\varphi_S(\nu_{\mathcal{F}}, \mathcal{F})$ and call it localization of $\varphi(\nu_{\mathcal{F}})$ by the foliation $\mathcal{F}$ at S.

If S is compact, as in Sect. 1.6.2, we have the residue $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S)$ (also called the Baum–Bott residue). We sometimes abbreviate it as $\mathrm{Res}_\varphi(\mathcal{F}, S)$. Moreover, if S has a finite number of connected components $(S_\lambda)_\lambda$, we have the residue $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S_\lambda)$ for each λ.

Let R_λ be a real $2m$-dimensional manifold with C^∞ boundary in U_1 containing S_λ in its interior and disjoint from the other components. Let also $R_{0\lambda} = -\partial R_\lambda$. Then the residue $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S)$ is a complex number given by

$$\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S_\lambda) = \int_{R_\lambda} \varphi(\nabla_1^\bullet) + \int_{R_{0\lambda}} \varphi(\nabla_0^\bullet, \nabla_1^\bullet). \tag{6.2.4}$$

The above residues are also originally introduced in [13, 14], where the residues are defined using forms with compact support.

From the above argument and Theorem 1.6.5, we have the following theorem.

Theorem 6.2.5. *In the above situation,*

(1) *For each connected component S_λ of $S(\mathcal{F})$, we have a well-defined residue $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S_\lambda)$, given by (6.2.4).*

(2) *If M is compact,*

$$\sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S_\lambda) = \int_M \varphi(\nu_{\mathcal{F}}).$$

Remark 6.2.2. 1. Suppose there is a generator v of $\mathcal{F}$ in a neighborhood of S_λ. Then, we may show ([156, Ch.III, Remark 7.7.1]) that

$$\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S_\lambda) = \mathrm{Res}_\varphi(v, TM; S_\lambda),$$

where the right hand side is the residue in Sect. 6.1. In particular, if S_λ consists of a point p, then $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; p)$ is given by the formula in Theorem 6.1.3.

If $\mathcal{F}$ is generated by a global vector field, then F is trivial and the results reduce to those in Sect. 6.1

2. Note that, in general, the bundle TM_0 or F_0 does not admit F_0-actions and that the bundle N_{F_0} does not extend to a bundle on M. Thus the residue $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}; S)$ is not of the type considered in the first part of Sect. 6.3.4 or Sect. 6.3.3 below.

3. The bundle map $F \to TM$ induces an injective map on the sheaf level with quotient $\mathcal{N}_{\mathcal{F}}$, which is the normal sheaf of the foliation. We have $\varphi(\nu_{\mathcal{F}}) = \varphi(\mathcal{N}_{\mathcal{F}})$. The residue theory for general singular foliation is developed using the sheaf $\mathcal{N}_{\mathcal{F}}$ ([14, 156]).
4. See [147] for the case of manifolds with boundary.

6.3 Residues of Holomorphic Vector Fields on Singular Varieties

Let M be a complex manifold of dimension $m = n + k$ and V an analytic variety of pure dimension n in M. If there is a holomorphic vector field, or more generally a one-dimensional singular foliation, leaving V invariant, we have three kinds of residues which we describe in the following subsections. The residue at an isolated singularity is expressed in terms of Grothendieck residue relative to the subvariety V.

6.3.1 Grothendieck Residues Relative to a Subvariety

Let $\widehat{U}$ be a neighborhood of 0 in $\mathbb{C}^{n+k}$ and V a subvariety of dimension n in $\widehat{U}$ which contains 0 as at most an isolated singular point. Also, let $f_1, \dots, f_n$ be holomorphic functions on $\widehat{U}$ and $V(f_1, \dots, f_n)$ the variety defined by them. We assume that $V(f_1, \dots, f_n) \cap V = \{0\}$. For a holomorphic n-from ω on $\widehat{U}$, the Grothendieck residue relative to V is defined by (e.g., [156, Ch.IV, 8])

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\Gamma} \frac{\omega}{f_1 \cdots f_n},$$

where Γ is the n-cycle in V given by

$$\Gamma = \{\, q \in \widehat{U} \cap V \mid |f_i(q)| = \varepsilon_i, \quad i = 1, \dots, n \,\}$$

for small positive numbers ε_i. It is oriented so that $d\arg(f_1) \wedge \cdots \wedge d\arg(f_n) \geq 0$.

If $k = 0$, it reduces to the usual Grothendieck residue (cf. Sect. 1.6.5), in which case we omit the suffix V.

If V is a complete intersection defined by $h_1 = \cdots = h_k = 0$ in $\widehat{U}$, we have

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \operatorname{Res}_0 \begin{bmatrix} \omega \wedge dh_1 \wedge \cdots \wedge dh_k \\ f_1, \dots, f_n, h_1, \dots, h_k \end{bmatrix}.$$

In the subsequent subsections, the following theorem ([112, Lemma 3]) is useful for expressing the residue in terms of the Grothendieck residue as above.

Theorem 6.3.1. *(Existence of a good coordinate system) Let* $(V, 0)$ *be a complete intersection in* $\mathbb{C}^{n+k}$ *and* $\widehat{v}$ *a holomorphic vector field on a neighborhood of* 0 *in* $\mathbb{C}^{n+k}$*. Suppose that* $S(\widehat{v}) \cap V = \{0\}$*. Then there exists a local coordinate system* $(z_1, \dots, z_{n+k})$ *about* 0 *such that, if we express* $\widehat{v}$ *as*

$$\widehat{v} = \sum_{i=1}^{n+k} a_i(z_1, \dots, z_{n+k}) \frac{\partial}{\partial z_i}$$

and if we denote by $(h_1, \dots, h_k)$ *a system of defining functions of* V *near* 0*, the set of common zeros of the holomorphic functions* $a_1, \dots, a_n, h_1, \dots, h_k$ *consists only of* 0*.*

Remark 6.3.1. In the above, it is not necessary to assume that $\widehat{v}$ is tangent to V_{reg}.

6.3.2 Residues for the Ambient Tangent Bundle (Generalized Variation)

Let M and V be as before and let $\widehat{v}$ be a holomorphic vector field on a neighborhood of V. We say that $\widehat{v}$ *leaves* V *invariant* (or is *logarithmic* for V, cf. [136]), if $\widehat{v}$ is tangent to the nonsingular part V_{reg} of V. Suppose $\widehat{v}$ is such a vector field and let v be the vector field on V_{reg} induced from $\widehat{v}$. We set $S = S(v, V) = S(v) \cup \mathrm{Sing}(V)$. Then $TM|_{V_0}$, $V_0 = V \setminus S$ is a v-bundle with the action

$$\alpha_v : A^0(V_0, TM|_{V_0}) \longrightarrow A^0(V_0, TM|_{V_0})$$

given as follows. For $w \in A^0(V_0, TM|_{V_0})$, taking its extension $\widehat{w}$ to some neighborhood of V_0, we set

$$\alpha_v(w) = [\widehat{v}, \widehat{w}]|_{V_0}. \tag{6.3.2}$$

Then it does not depend on the extension $\widehat{w}$ and defines a holomorphic action.

Let $\widehat{U}_0$, $\widehat{U}_1$, $\widehat{\mathcal{U}}$ and $\widehat{U} = \widehat{U}_0 \cup \widehat{U}_1$ be as in Sect. 5.3. Because of this action, the characteristic class $\varphi(TM)$ in $H^{2n}_D(\widehat{\mathcal{U}})$ is lifted to a class $\varphi(TM, \widehat{v})$ in $H^{2n}_D(\widehat{\mathcal{U}}, \widehat{U}_0)$. Moreover, if S is compact, this localization give rise to a residue, which we denote by $\mathrm{Res}_\varphi(\widehat{v}, TM|_V; S)$ instead of $\mathrm{Res}_\varphi(v, TM|_V; S)$, since it depends on the extension $\widehat{v}$ of v. This residue is introduced in [112, 3,Example 2], see also [113].

From Theorem 5.3.7, we have the following theorem.

Theorem 6.3.3. *In the above situation, for each connected component* S_λ *of* $S(v,V)$ *and a homogeneous symmetric polynomial* φ *of degree* n*, we have the residue* $\mathrm{Res}_\varphi(\widehat{v}, TM|_V; S_\lambda)$ *and, if* V *is compact,*

$$\sum_\lambda \mathrm{Res}_\varphi(\widehat{v}, TM|_V; S_\lambda) = \int_V \varphi(TM).$$

Now we assume that S consists of an isolated point p, $\widehat{v}$ is defined near p and that V is a complete intersection near p. Let $(z_1,\dots,z_{n+k})$ be a local coordinate system on M near p and let A be the Jacobian matrix of $(a_1,\dots,a_{n+k})$ with respect to $(z_1,\dots,z_{n+k})$. Then we have the following formula (cf. [156, Ch. IV, Theorem 5.3]).

Theorem 6.3.4. *In the above situation, if* $(z_1,\dots,z_{n+k})$ *is a coordinate system as in Theorem 6.3.1,*

$$\mathrm{Res}_\varphi(\widehat{v}, TM|_V; p) = \mathrm{Res}_p \begin{bmatrix} \varphi(A) dz_1 \wedge \cdots \wedge dz_n \\ a_1, \dots, a_n \end{bmatrix}_V .$$

Now suppose more generally that we have a one-dimensional foliation $\mathcal{F}$ on M leaving V invariant, *i.e.*, every vector field locally defining $\mathcal{F}$ leaves V invariant. We denote by $\mathcal{F}_V$ the one-dimensional foliation induced on the nonsingular part of V and set $S = S(\mathcal{F}, V) = (S(\mathcal{F}) \cap V)) \cup \mathrm{Sing}(V)$ and $V_0 = V \setminus S(\mathcal{F}, V)$. We also set $F_V = F|_V$ and $F_{V_0} = F|_{V_0}$. Recall that on M_0 there is a vector bundle N_{F_0} and the exact sequence (6.2.2). The action of F_0 on N_{F_0} induces an action of F_{V_0} on $N_{F_0}|_{V_0}$. Thus the restriction to V of characteristic class $\varphi(\nu_\mathcal{F})$ of the virtual normal bundle $\nu_\mathcal{F} = TM - F$ of the foliation $\mathcal{F}$ for a homogeneous symmetric polynomial φ of degree n is localized at the singular set S. Moreover, if S is compact, we have the residue $\mathrm{Res}_\varphi(\mathcal{F}, \nu_\mathcal{F}|_V; S)$.

From Theorem 5.3.7, we have the following

Theorem 6.3.5. *In the above situation, for each connected component* S_λ *of* $S(\mathcal{F},V)$ *and a homogeneous symmetric polynomial* φ *of degree* n*, we have the residue* $\mathrm{Res}_\varphi(\mathcal{F}, \nu_\mathcal{F}|_V; S_\lambda)$ *and, if* V *is compact,*

$$\sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}, \nu_\mathcal{F}|_V; S_\lambda) = \int_V \varphi(\nu_\mathcal{F}).$$

Remark 6.3.2. 1. Suppose there is a vector field $\widehat{v}$ defining $\mathcal{F}$ on a neighborhood $\widehat{U}$ of S in M. Then we have

$$\mathrm{Res}_\varphi(\mathcal{F}, \nu_\mathcal{F}|_V; S) = \mathrm{Res}_\varphi(\widehat{v}, TM|_V; S).$$

In particular, if S consists of a point p, then $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}|_V; p)$ is given by the formula in Theorem 6.3.4. If $\mathcal{F}$ is generated by a global vector field $\widehat{v}$, then F_V is trivial and the results reduce to those of $\widehat{v}$ for $TM|_V$.

2. If $n = k = 1$, $V = C$ is a (possibly singular) curve in a (nonsingular) complex surface M. We have the residue essentially only for $\varphi = c^1$ and $S(\mathcal{F}, C)$ consists of isolated points. The residue $\mathrm{Res}_{c^1}(\mathcal{F}, \nu_{\mathcal{F}}|_C; p)$ coincides with the "variation" $\mathrm{Var}(\mathcal{F}, C; p)$ introduced in [93] and Theorem 6.3.5 gives a formula for the sum of the variations, which is proved in [93] by a different approach.

6.3.3 Residues for the Normal Bundle (Residues of Type Camacho–Sad)

The residues in this subsection are introduced in [112], generalizing the ones already known in various special cases (cf. Remark 6.3.3 below).

In this section, we assume that V is an LCI of dimension n defined by a section of a holomorphic vector bundle N over M. Recall that $N|_{V_{\mathrm{reg}}} = N_{V_{\mathrm{reg}}}$, the normal bundle of the regular part V_{reg}. We set $N_V = N|_V$. Let $\widehat{v}$ be a holomorphic vector field on a neighborhood of V in M which leaves V invariant. Letting v be the vector field on (the regular part of) V induced by $\widehat{v}$, we define the singular set $S = S(v, V)$ as before. Then N_{V_0}, $V_0 = V \setminus S(v, V)$, is a v-bundle with the action

$$\alpha_v : A^0(V_0, N_{V_0}) \longrightarrow A^0(V_0, N_{V_0})$$

given as follows. Recall the exact sequence

$$0 \longrightarrow TV_{\mathrm{reg}} \longrightarrow TM|_{V_{\mathrm{reg}}} \xrightarrow{\pi} N_{V_{\mathrm{reg}}} \longrightarrow 0.$$

For $\nu \in A^0(V_0, N_{V_0})$, we may write $\nu = \pi(\widehat{w}|_{V_0})$ with $\widehat{w}$ a section of TM in a neighborhood of V_0. Then we set

$$\alpha_v(\nu) = \pi([\widehat{v}, \widehat{w}]|_{V_0}). \qquad (6.3.6)$$

Then it does not depend on the choice of $\widehat{w}$ and defines a holomorphic action. For a symmetric homogeneous polynomial φ of degree n, we have the localization of $\varphi(N_V)$ at S, which we denote by $\varphi(N_V, \widehat{v})$, instead of $\varphi(N_V, v)$, since it depends on the extension $\widehat{v}$ of v. Moreover, if S is compact, we have the residue $\mathrm{Res}_\varphi(\widehat{v}, N_V; S)$. From Theorem 5.3.7, we have the following

Theorem 6.3.7. *In the above situation, for each connected component S_λ of $S(v, V)$ and a homogeneous symmetric polynomial φ of degree n, we have the residue* $\mathrm{Res}_\varphi(\widehat{v}, N_V; S_\lambda)$ *and, if V is compact,*

$$\sum_{\lambda} \mathrm{Res}_{\varphi}(\widehat{v}, N_V; S_\lambda) = \int_V \varphi(N).$$

Now we assume that S consists of an isolated point p. If we let $f = (f_1, \dots, f_k)$ be a system of defining functions of V near p, the condition that V is invariant by $\widehat{v}$ is expressed as

$$\widehat{v}(f_i) = \sum_{i=1}^{k} k_{ij} f_j,$$

for some holomorphic functions k_{ij} ([136], [38]). Let $K = (k_{ij})$ and φ a homogeneous symmetric polynomial of degree n. Then we have the following formula (cf. [156, Ch. IV, Theorem 6.3]).

Theorem 6.3.8. *In the above situation, if* $(z_1, \dots, z_{n+k})$ *denotes a coordinate system as in Theorem 6.3.1,*

$$\mathrm{Res}_{\varphi}(\widehat{v}, N_V; p) = \mathrm{Res}_p \begin{bmatrix} \varphi(K) dz_1 \wedge \cdots \wedge dz_n \\ a_1, \dots, a_n \end{bmatrix}_V .$$

Now suppose we have a one-dimensional foliation $\mathcal{F}$ on M leaving V invariant. We denote by $\mathcal{F}_V$ the one-dimensional foliation induced on V_{reg} and set $S = S(\mathcal{F}, V) = (S(\mathcal{F}) \cap V)) \cup \mathrm{Sing}(V)$ and $V_0 = V \setminus S$ as before. Let F be the tangent bundle of $\mathcal{F}$ and set $F_{V_0} = F|_{V_0}$ as in Sect. 6.3.2. Then we see that there is an action of F_{V_0} on N_{V_0} and, for a homogeneous symmetric polynomial φ of degree n, there is a localization $\varphi(N_V, \mathcal{F})$ of $\varphi(N_V)$ by $\mathcal{F}$. Moreover, if S is compact, we have the residue $\mathrm{Res}_{\varphi}(\mathcal{F}, N_V; S)$. From Theorem 5.3.7, we have the following

Theorem 6.3.9. *In the above situation, for each connected component* S_λ *of* $S(\mathcal{F}, V)$ *and a homogeneous symmetric polynomial* φ *of degree* n*, we have the residue* $\mathrm{Res}_{\varphi}(\mathcal{F}, N_V; S_\lambda)$*, and if* V *is compact,*

$$\sum_{\lambda} \mathrm{Res}_{\varphi}(\mathcal{F}, N_V; S_\lambda) = \int_V \varphi(N).$$

Remark 6.3.3. 1. Suppose there is a vector field $\widehat{v}$ defining $\mathcal{F}$ on a neighborhood $\widehat{U}$ of S in M. Then we have

$$\mathrm{Res}_{\varphi}(\mathcal{F}, N_V; S) = \mathrm{Res}_{\varphi}(\widehat{v}, N_V; S).$$

In particular, if S consists of a point p, then $\mathrm{Res}_{\varphi}(\mathcal{F}, N_V; p)$ is given by the formula in Theorem 6.3.8. If $\mathcal{F}$ is generated by a global vector field $\widehat{v}$, then F is trivial and the results reduce to those of $\widehat{v}$ for N_V.

2. If $n = k = 1$, $V = C$ is a (possibly singular) curve in a (nonsingular) complex surface M. We have the residue essentially only for $\varphi = c^1$ and $S(\mathcal{F}, C)$ consists of isolated points. If C is nonsingular at p, the residue $\mathrm{Res}_{c^1}(\mathcal{F}, N_C; p)$ coincides with the index defined by C. Camacho and P. Sad in [42] and, in general, it coincides with the one defined by T. Suwa in [154], which is also equal to the one defined by A. Lins Neto in [114], provided C is locally irreducible at p. Also, if C is compact, we have, by Theorem 6.3.9,

$$\sum_{p \in S(\mathcal{F}, C)} \mathrm{Res}_{c^1}(\mathcal{F}, N_C; p) = \int_C c^1(N) = C^2,$$

where C^2 denotes the self-intersection number of C in X. This is the formula proved in [42] when C is nonsingular and is the one proved in [154] in general. Equivalent formulas are obtained for $X = \mathbb{CP}^2$ in [114]. We call $\mathrm{Res}_{c^1}(\mathcal{F}, N_C; p)$ the Camacho–Sad index and also denote it by $\mathrm{Ind}_{\mathrm{CS}}(\mathcal{F}, C; p)$.

3. If V is nonsingular, Theorems 6.3.7, 6.3.8 and 6.3.9 reduce to the ones in [110]. In particular, the formula in 6.3.8 becomes as follows. If p is a nonsingular point of V, we may take a local coordinate system $(z_1, \dots, z_{n+k})$ so that $f_1 = z_{n+1}, \dots, f_k = z_{n+k}$ define V near p. Then the invariance condition for V by $\widehat{v}$ is that $a_{n+1}, \dots, a_{n+k}$ vanish on V. Thus $(z_1, \dots, z_{n+k})$ is a good coordinate system in the sense of Theorem 6.3.1. On the other hand, k_{ij} and $\partial a_{n+i}/\partial z_{n+j}$ are equal on V.

Let us denote by A the Jacobian matrix $\partial(a_1, \dots, a_{n+k})/\partial(z_1, \dots, z_{n+k})$ and by A_1 and A_2 the Jacobian sub-matrices

$$\begin{aligned} A_1 &= \partial(a_1, \dots, a_n)/\partial(z_1, \dots, z_n) \\ A_2 &= \partial(a_{n+1}, \dots, a_{n+k})/\partial(z_{n+1}, \dots, z_{n+k}). \end{aligned}$$

We recover the formula

$$\mathrm{Res}_\varphi(\mathcal{F}, N_V; p) = \mathrm{Res}_p \begin{bmatrix} \varphi(A_2) dz_1 \wedge \cdots \wedge dz_n \\ a_1, \dots, a_n \end{bmatrix}_V$$

in [110, Théorème 1].

4. Theorem 6.3.9 is generalized to the case of higher dimensional singular foliations on M (cf. B. Gmira [67], J.-P. Brasselet in ([115], Remark 1), A. Lins Neto [115], D. Lehmann [110] and for a general setting T. Suwa [156]).

5. In [1], M. Abate proved a Camacho–Sad type index theorem for fixed curves of holomorphic self-maps of complex surfaces and used it to prove the existence of a parabolic curve at a fixed point of a holomorphic self-map of the complex plane. Then it was realized by M. Abate, F. Bracci and F. Tovena that the index theorem can be proved and generalized in the framework of the residue theory for holomorphic foliations as explained in this section, see

[2,21,24]. Furthermore, in [3] the residue theories for holomorphic maps and for foliations are unified and much generalized using a version of vanishing theorem finer than the one we expect to have by assuming the involutiveness.

In [22,23], the residue theory for fixed subvarieties of holomorphic self-map of certain singular varieties is developed. It is used to prove the existence of a parabolic curve at a fixed point of a holomorphic self-map of surfaces with certain type of singularity.

6.3.4 Residues for the Virtual Tangent Bundle (Singular Baum–Bott)

Let M, V, $\widehat{v}$, v and $S = S(v, V)$ be as in the previous section. Recall that the virtual tangent bundle of V is defined by $\tau_V = TM|_V - N_V$. From the actions of v on the bundles $TM|_V$ and N_V given by (6.3.2) and (6.3.6), respectively, we have the localization $\varphi(\tau_V, \widehat{v})$ of $\varphi(\tau_V)$ for a symmetric homogeneous polynomial of degree n. Moreover, if S is compact, we have the residue $\mathrm{Res}_\varphi(\widehat{v}, \tau_V; S_\lambda)$ for each connected component S_λ of S. From Theorem 5.3.7, we have the following theorem ([156, Ch.IV, Theorem 7.1]).

Theorem 6.3.10. *In the above situation, for each connected component* S_λ *of* $S(v, V)$ *and a homogeneous symmetric polynomial* φ *of degree* n*, we have the residue* $\mathrm{Res}_\varphi(\widehat{v}, \tau_V; S_\lambda)$*, and if* V *is compact,*

$$\sum_\lambda \mathrm{Res}_\varphi(\widehat{v}, \tau_V; S_\lambda) = \int_V \varphi(\tau).$$

Now we assume that S_λ consists of an isolated point p and that we have a vector field $\widehat{v}$ defined in a neighborhood of p in M. For a homogeneous symmetric polynomial φ of degree n, we write $\varphi = \sum_\ell \varphi_\ell \cdot \varphi'_\ell$ so that $\varphi(\tau) = \sum_\ell \varphi_\ell(TM) \cdot \varphi'_\ell(N)$ and set $\varphi(H) := \sum_\ell \varphi_\ell(A) \cdot \varphi'_\ell(K)$. Then we have the following formula (cf. [156, Ch. IV, Theorem 7.2]).

Theorem 6.3.11. *In the above situation, if* $(z_1, \dots, z_{n+k})$ *denotes a coordinate system as in Theorem 6.3.1,*

$$\mathrm{Res}_\varphi(\widehat{v}, \tau_V; p) = \mathrm{Res}_p \begin{bmatrix} \varphi(H) dz_1 \wedge \cdots \wedge dz_n \\ a_1, \dots, a_n \end{bmatrix}_V .$$

In particular, if $\varphi = c^n$, the above formula coincides with the formula in Theorem 5.7.1 (cf. Proposition 6.3.1 below). These residues together with the formulas as above are given in [111].

Remark 6.3.4. If p is a regular point of V, then we may take a coordinate system $(z_1, \dots, z_{n+k})$ about p as in Remark 6.3.3, 3. Then, on V, $K = A_2$

and $\partial a_{p+i}/\partial z_j = 0$ for $i = 1, \dots, k$ and $j = 1, \dots, n$. Hence the formula in Theorem 6.3.11 reduces to

$$\mathrm{Res}_\varphi(\widehat{v}, \tau_V; p) = \mathrm{Res}_p \begin{bmatrix} \varphi(A_1) dz_1 \wedge \cdots \wedge dz_n \\ a_1, \dots, a_n \end{bmatrix}_V,$$

which is the Baum–Bott residue (Theorem 6.1.3) on V.

In fact, for this residue, we only need a vector field on the nonsingular part of V. Thus, suppose we are given a holomorphic vector field v on the regular part of V. We denote by $S = S(v, V)$ the union of $\mathrm{Sing}(V)$ and the singular set $S(v)$ of v and set $V_0 = V \setminus S$. Let U, U_0 and $\widehat{U}_1$ be as before. On V_0, TV_0 is a v-bundle with the action given in Sect. 6.1. This action defines a localization $\varphi(\tau_V, v)$ of $\varphi(\tau_V)$ at S. If S is compact, we have the residue $\mathrm{Res}_\varphi(v, \tau_V; S_\lambda)$ for each connected component S_λ of S.

Note that, if we have an extension $\widehat{v}$ of v, we have

$$\mathrm{Res}_\varphi(\widehat{v}, \tau_V; S_\lambda) = \mathrm{Res}_\varphi(v, \tau_V; S_\lambda).$$

Note also that, if S_λ is in the regular part of V, then, we have

$$\mathrm{Res}_\varphi(v, \tau_V; S_\lambda) = \mathrm{Res}_\varphi(v, TV; S_\lambda),$$

the residue in Sect. 6.1.

Theorem 6.3.12. *Let v be a holomorphic vector field on the regular part V_{reg} of a variety V of dimension n as above and φ a homogeneous symmetric polynomial of degree n. For each connected component S_λ of $S(v, V)$, we have the residue $\mathrm{Res}_\varphi(v, \tau_V; S_\lambda)$, and if V is compact,*

$$\sum_\lambda \mathrm{Res}_\varphi(v, \tau_V; S_\lambda) = \int_V \varphi(\tau).$$

We also have the following (cf. [156, Ch.IV, Proposition 7.8])

Proposition 6.3.1. *In the above situation,*

$$\mathrm{Res}_{c^n}(v, \tau_V; S_\lambda) = \mathrm{Ind}_{\mathrm{Vir}}(v, S_\lambda).$$

Now suppose we have a one-dimensional foliation $\mathcal{F}$ on M leaving V invariant. We denote by $\mathcal{F}_V$ the one-dimensional foliation induced on V_{reg} and set $S = S(\mathcal{F}, V) = (S(\mathcal{F}) \cap V)) \cup \mathrm{Sing}(V)$, $M_0 = M \setminus S(\mathcal{F})$ and $V_0 = V \setminus S$. Let F be the tangent bundle of $\mathcal{F}$ and set $F_0 = F|_{M_0}$ and $F_{V_0} = F|_{V_0}$ as before. Recall that $N_{F_0}|_{V_0}$ and N_{V_0} are F_{V_0}-bundles with the actions given in Sects. 6.3.2 and 6.3.3. If we set $N_{F_{V_0}} = TV_0/F_{V_0}$, it is also an F_{V_0}-bundle with the action given in Sect. 6.2.

We try to compute the restriction to V of the class $\varphi(TM - N - F)$ of the virtual bundle $TM - N - F = \nu_{\mathcal{F}} - N = \tau - F$ for a homogeneous symmetric polynomial φ of degree n. In fact there are two ways to localize the class. Namely, we have the residue $\mathrm{Res}_\varphi(\mathcal{F}, (\nu_{\mathcal{F}} - N)|_V; S)$ of $\mathcal{F}$ on $(\nu_{\mathcal{F}} - N)|_V$, which come from the actions of F_{V_0} on $N_{F_0}|_{V_0}$ and N_{V_0}, and the residue $\mathrm{Res}_\varphi(\mathcal{F}_V, (\tau - F)|_V; S)$ of $\mathcal{F}_V$ on $(\tau - F)|_V$, which comes from the action of F_{V_0} on $N_{F_{V_0}}$. However, it can be shown that in fact the two residues are equal ([156, Ch.IV, (7.11)]):

$$\mathrm{Res}_\varphi(\mathcal{F}, (\nu_{\mathcal{F}} - N)|_V; S) = \mathrm{Res}_\varphi(\mathcal{F}_V, (\tau - F)|_V; S). \tag{6.3.13}$$

Denoting by $\nu_{\mathcal{F}_V}$ the restriction $(TM-N-F)|_V = (\nu_{\mathcal{F}}-N)|_V = (\tau-F)|_V$, we denote the above residue by $\mathrm{Res}_\varphi(\mathcal{F}_V, \nu_{\mathcal{F}_V}; S)$. We quote the following theorem from [156, Ch.IV, Theorem 7.12].

Theorem 6.3.14. *In the above situation, for each connected component S_λ of $S(\mathcal{F}, V)$ and a homogeneous symmetric polynomial φ of degree n, the residue* $\mathrm{Res}_\varphi(\mathcal{F}_V, \nu_{\mathcal{F}_V}; S_\lambda)$ *is defined, and if V is compact,*

$$\sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}_V, \nu_{\mathcal{F}_V}; S_\lambda) = \int_V \varphi(\tau - F).$$

Remark 6.3.5. 1. Suppose there is a vector field $\widehat{v}$ defining $\mathcal{F}$ on a neighborhood $\widehat{U}$ of S in M. Then we have

$$\mathrm{Res}_\varphi(\mathcal{F}, (\nu_{\mathcal{F}} - N)|_V; S) = \mathrm{Res}_\varphi(\widehat{v}, \tau_V; S).$$

In particular, if S consists of a point p, the residue

$$\mathrm{Res}_\varphi(\mathcal{F}_V, \nu_{\mathcal{F}_V}; p) = \mathrm{Res}_\varphi(\mathcal{F}, (\nu_{\mathcal{F}} - N)|_V; p)$$

is given by the formula in Theorem 6.3.11. If $\mathcal{F}$ is generated by a global vector field $\widehat{v}$, then F is trivial and the results reduce to those of $\widehat{v}$ for τ_V.

2. Note that, in general, the bundle $TM|_{V_0}$ or F_{V_0} does not admit F_{V_0}-actions and that the bundle $N_{F_0}|_{V_0}$ or $N_{F_{V_0}}$ does not extend to a bundle on V.

3. If V is nonsingular, the virtual bundle $\nu_{\mathcal{F}_V}$ is equivalent to $TV - F_V$ and the above theory reduces to the one in Sect. 6.2.

4. Theorem 6.3.14 is generalized to the case of higher dimensional singular foliations on M (see [156, Ch.VI]).

From Proposition 6.3.1, (6.3.13) and Remark 6.3.5, 1, we have the following:

Proposition 6.3.2. *Let M, V and $\mathcal{F}$ be as above and let S be a compact connected component of $S(\mathcal{F}, V)$. Suppose there is a vector field $\widehat{v}$ defining $\mathcal{F}$*

on a neighborhood $\widehat{U}$ *of* S *in* M *and denote by* v *the vector field on* $\widehat{U} \cap V_{\mathrm{reg}}$ *induced from* $\widehat{v}$*. Then* $\mathrm{Ind}_{\mathrm{Vir}}(v, S)$ *does not depend on the choice of such a vector field* $\widehat{v}$ *and is an invariant of* $\mathcal{F}_V$*.*

In particular, considering the case $S = \{p\}$, from Theorem 5.5.1, we have:

Corollary 6.3.1. *In the above situation,* $\mathrm{Ind}_{\mathrm{GSV}}(v, p)$ *is an invariant of the foliation* $\mathcal{F}_V$*.*

In view of the above, we denote $\mathrm{Ind}_{\mathrm{GSV}}(v, p)$ by $\mathrm{Ind}_{\mathrm{GSV}}(\mathcal{F}_V, p)$. The following theorem is a consequence of Theorems 5.5.1, 5.6.3 and 6.3.14 and Proposition 6.3.1.

Theorem 6.3.15. *Let* M*,* V *and* $\mathcal{F}$ *be as above. Assume that* V *is compact and that* $S(\mathcal{F}, V)$ *consists of a finite number of points* $p_1, \dots, p_r$*. Then*

$$\sum_{i=1}^{r} \mathrm{Ind}_{\mathrm{GSV}}(\mathcal{F}_V, p_i) = \int_V c^n(\tau - F).$$

Remark 6.3.6. 1. If $n = k = 1$, $V = C$ is a (possibly singular) curve in a (nonsingular) complex surface. We have $c^1(\tau - F) = c^1(\tau) - c^1(F)$ and $\int_C c^1(\tau) =: \chi'(C)$, the virtual Euler–Poincaré characteristic of C. Hence the above formula becomes

$$\sum_{i=1}^{r} \mathrm{Ind}_{\mathrm{GSV}}(\mathcal{F}_C, p_i) = \chi'(C) - \int_C c^1(F),$$

which is proved independently in [93] and by Brunella [39], by different approaches.

There are some interesting relations among the residues in Sects. 6.3.2–6.3.4. A general formula is given in [156, Ch.IV, 7]. Here we give a formula in the case $n = k = 1$. Thus $V = C$ is a possibly singular curve in (a nonsingular) surface M. In this case, we have

$$\mathrm{Ind}_{\mathrm{GSV}}(\mathcal{F}_C, p) = \mathrm{Var}(\mathcal{F}, C; p) - \mathrm{Ind}_{\mathrm{CS}}(\mathcal{F}, C; p).$$

There are explicit examples of the residues in [156, Ch.IV, 8]. Example 5.7.1 is one of them.

Chapter 7
The Homological Index and Algebraic Formulas

Abstract We have already defined and studied several indices of vector fields on singular varieties, each of them being related to some property of the index of Poincaré–Hopf, or to some extension of the tangent bundle to the case of singular varieties. There is another line of research with remarkable works by various authors, that originates in the well-known fact (cf. Example 1.6.2) that for a holomorphic vector field v in $\mathbb{C}^n$ with an isolated singularity at 0, the local Poincaré–Hopf index satisfies:

$$\mathrm{Ind}_{\mathrm{PH}}(v, 0) \,=\, \dim \mathcal{O}_{\mathbb{C}^n,0}/(a_1, \cdots, a_n), \tag{7.0.1}$$

where $(a_1, \cdots, a_n)$ is the ideal generated by the components of v. In the real analytic setting, the equivalent statement is given by the formula of Eisenbud–Levin–Khimshiashvili, expressing the local Poincaré–Hopf index through the signature of a certain quadratic form.

These facts motivated the search for algebraic formulas for indices of vector fields on singular varieties. A major contribution in this direction was given by V. I. Arnold for gradient vector fields. There are also significant contributions by various authors, such as X. Gómez-Mont, S. Gusein-Zade, W. Ebeling and others.

In this chapter we give a glance of some of the research in this direction, and we refer to the literature for more on that topic. We discuss first the homological index for holomorphic vector fields, introduced by X. Gómez-Mont and further studied by himself in collaboration with Ch. Bonatti, P. Mardešić, L. Giraldo, H.-C. G. von Bothmer and W. Ebeling. In the last section of this chapter we discuss briefly the Eisenbud–Levin–Khimshiashvili formula for the index of real analytic vector fields, and its generalization to singular varieties.

The homological index has the important property of being defined for holomorphic vector fields on arbitrary complex analytic isolated singularity germs $(V, 0)$. When the germ $(V, 0)$ is a complete intersection, the homological index coincides with the GSV-index, by [17, 68].

If we now let V be a compact complex variety with isolated singularities, one has a well-defined notion of the *total homological index* for holomorphic

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vector fields on V with isolated singularities, defined in the usual way. This total index is independent of the choice of vector field, being therefore an invariant of V. If V is a local complete intersection as in Sect. 5.4, then the corresponding global invariant is the 0-degree Fulton–Johnson class of V (see in Chap. 11).

It would be interesting to know what the homological index measures for singular germs and varieties which are not local ICIS. This is related with extending the notion of Milnor number to isolated singularity germs which are not complete intersections (see Chap. 9).

7.1 The Homological Index

The basic references for this section are the articles by Gómez-Mont and various co-authors, see [17, 62–65, 68–70]. See also [7, 92, 97].

An algebraic formula for the index of holomorphic vector fields on singular varieties was given in [71], inspired by (7.0.1), but that formula applies only under very stringent conditions: for holomorphic vector fields on a hypersurface germ V which are tangent to the fibers of a defining function f of V.

Using the fact that in the classical case, when the ambient space is smooth, the Poincaré–Hopf local index can be interpreted as the Euler-characteristic of a certain Koszul complex (see [75]), Gómez-Mont introduced in [68] a notion of the homological index of holomorphic vector fields. Let us explain this invariant.

Let $(V,0) \subset (\mathbb{C}^m,0)$ be a germ of a complex analytic variety of pure dimension n, which is regular on $V \setminus \{0\}$. So V is either regular at 0 or else it has an isolated singular point at the origin. A vector field v on $(V,0)$ can always be defined as the restriction to V of a vector field $\widehat{v}$ in the ambient space which is tangent to $V \setminus \{0\}$; v is holomorphic if $\widehat{v}$ can be chosen to be holomorphic. So we may write v as $v = (a_1, \cdots, a_m)$ where the a_i are restriction to V of holomorphic functions on a neighborhood of 0 in $(\mathbb{C}^m,0)$.

It is worth noting that given any space V as above, there are always holomorphic vector fields on V with an isolated singularity at 0. This nontrivial fact is indeed a weak form of a stronger result ([16, p. 19]): in the space $\Theta(V,0)$ of germs of holomorphic vector fields on V at 0, those having an isolated singularity form a connected, dense open subset $\Theta_0(V,0)$. Essentially the same result implies also that every $v \in \Theta_0(V,0)$ can be extended to a germ of holomorphic vector field in $\mathbb{C}^m$ with an isolated singularity, though it can also be extended with a singular locus of dimension more that 0, a fact that may be useful for explicit computations (c.f. the last part of the following section).

A (germ of) holomorphic j-form on V at 0 means the restriction to V of a holomorphic j-form on a neighborhood of 0 in $\mathbb{C}^m$; two such forms in $\mathbb{C}^m$ are equivalent if their restrictions to V coincide on a neighborhood of

$0 \in V$. We denote by $\Omega^j_{V,0}$ the space of all such forms (germs); these are the Kähler differential forms on V at 0. So $\Omega^0_{V,0}$ is the local structure ring $\mathcal{O}_{(V,0)}$ of holomorphic functions on V at 0 and each $\Omega^j_{V,0}$ is an $\Omega^0_{V,0}$-module. Notice that if the germ of V at 0 is determined by $(f_1, \cdots, f_k)$ then one has:

$$\Omega^j_{V,0} := \Omega^j_{\mathbb{C}^m,0}/(f_1\Omega^j_{\mathbb{C}^m,0} + df_1 \wedge \Omega^{j-1}_{\mathbb{C}^m,0}, \dots, f_k\Omega^j_{\mathbb{C}^m,0} + df_k \wedge \Omega^{j-1}_{\mathbb{C}^m,0}), \quad (7.1.1)$$

where d is the exterior derivative.

Now, given a holomorphic vector field $\widehat{v}$ at $0 \in \mathbb{C}^m$ with an isolated singularity at the origin, and a differential form $\omega \in \Omega^j_{\mathbb{C}^m,0}$, we can always contract ω by v in the usual way, thus getting a differential form $i_v(\omega) \in \Omega^{j-1}_{\mathbb{C}^m,0}$. If $v = \widehat{v}|_V$ is tangent to V, then contraction is well defined at the level of differential forms on V at 0 and one gets a complex $(\Omega^\bullet_{V,0}, v)$:

$$0 \longrightarrow \Omega^n_{V,0} \longrightarrow \Omega^{n-1}_{V,0} \longrightarrow \cdots \longrightarrow \mathcal{O}_{V,0} \longrightarrow 0, \quad (7.1.2)$$

where the arrows are contraction by v and n is the dimension of V; of course one also has differential forms of degree $> n$, but those forms do not play a significant role here. We consider the homology groups of this complex:

$$H_j(\Omega^\bullet_{V,0}, v) = \mathrm{Ker}\,(\Omega^j_{V,0} \to \Omega^{j-1}_{V,0})/\mathrm{Im}\,(\Omega^{j+1}_{V,0} \to \Omega^j_{V,0})$$

The first observation in [68] is that if V is regular at 0, so that its germ at 0 is that of $\mathbb{C}^n$ at the origin, and if $v = (a_1, \cdots, a_n)$ has an isolated singularity at 0, then this is the usual Koszul complex. In that case, its homology groups vanish for $j > 0$, while

$$H_0(\Omega^\bullet_{V,0}, v) \cong \mathcal{O}_{\mathbb{C}^n,0}/(a_1, \cdots, a_n).$$

In particular the complex is exact if $v(0) \neq 0$. Since the contraction maps are $\mathcal{O}_{V,0}$-module maps, this implies that if V has an isolated singularity at the origin, then the homology groups of this complex are concentrated at 0, and they are finite dimensional because the sheaves of Kähler differentials on V are coherent. Hence it makes sense to define, for V a complex analytic germ with an isolated singularity at 0 and v a holomorphic vector field on V with an isolated singularity at 0:

Definition 7.1.1. The homological index $\mathrm{Ind}_{\mathrm{hom}}(v, 0; V)$ of the holomorphic vector field v on $(V, 0)$ is the Euler characteristic of the above complex:

$$\mathrm{Ind}_{\mathrm{hom}}(v, 0; V) = \sum_{i=0}^{n} (-1)^i h_i(\Omega^\bullet_{V,0}, v),$$

where $h_i(\Omega^{\bullet}_{V,0}, v)$ is the dimension of the corresponding homology group as a vector space over $\mathbb{C}$.

We recall that an important property of the Poincaré–Hopf local index is its stability under perturbations. This means that if we perturb v slightly in a neighborhood of an isolated singularity, then this zero of v may split into a number of isolated singularities of the new vector field v', such that the sum of indices of v' at these singular points equals the index of v. If the ambient space V has an isolated singularity at 0, then every vector field on V necessarily vanishes at 0, since in the ambient space the vector field defines a local flow with 0 as fixed point. Hence every perturbation of v producing a vector field tangent to V must also vanish at 0, but new singularities may arise with this perturbation. The homological index satisfies a stability under this type of perturbations (called the "Law of Conservation of Number" in [64, 68]):

Theorem 7.1.3. (Gómez-Mont [68, Theorem 1.2]) *For every holomorphic vector field v' on V sufficiently close to v one has:*

$$\mathrm{Ind}_{\mathrm{hom}}(v, 0; V) \;=\; \mathrm{Ind}_{\mathrm{hom}}(v', 0; V) \,+ \sum \mathrm{Ind}_{\mathrm{PH}}(v'),$$

where the sum on the right runs over the singularities of v' at regular points of V near 0.

This theorem is a special case of the following general theorem of [64]:

Theorem 7.1.4. (Law of Conservation of Number) *Let T and V be complex analytic spaces with T reduced and locally irreducible, and let $\pi : T \times V \to T$ be the projection to the first factor. Let $\mathcal{K}^*$ be a complex of $\mathcal{O}_{T\times V}$ coherent sheaves,*

$$0 \longrightarrow \mathcal{K}^n \longrightarrow \mathcal{K}^{n-1} \longrightarrow \cdots \longrightarrow \mathcal{K}^0 \longrightarrow 0,$$

where the sheaves $\mathcal{K}^j$ are $\mathcal{O}_T$-flat and the support of the homology sheaves $H^j(\mathcal{K}^*)$ is π-finite. Then, for every $(t_0, p_0) \in T \times V$ there are neighborhoods T' and V' of t_0 and p_0, respectively, such that for every $t \in T'$ we have:

$$\chi(\mathcal{K}^*_{t_0,p_0}) \;=\; \sum_{q\in V'} \chi(\mathcal{K}^*_{t,q}),$$

where χ denotes the Euler characteristic of the homology groups (vector spaces) of the corresponding complexes.

As mentioned before, it is proved in [17] that if the germ $(V, 0)$ is an ICIS then the homological and GSV indices coincide, a fact previously known only for hypersurface germs (see [68]). Theorem 7.1.3 plays a key-role for identifying these indices, as we explain below in the case of hypersurfaces.

Remark 7.1.1. 1. Given V and v as above, the Schwartz index of v at 0, introduced in Chap. 2, is also defined, and one can easily show, using Theorem 7.1.3, that the difference:

$$\mathrm{Ind}_{\mathrm{Sch}} - \mathrm{Ind}_{\mathrm{hom}}$$

is a constant that depends only of the space V and not on the choice of the vector field v. What is this constant?. We will see in Sect. 2 below that for hypersurface singularities this is the Milnor number (up to sign). By [17] the same statement holds for ICIS.

2. If V is smooth at 0 then the Poincaré–Hopf local index of every holomorphic vector field is necessarily ≥ 0. The analogous statement for the GSV-index is easily seen to be false if V is singular. For example, if V is the affine curve in $\mathbb{C}^2$ defined by the homogeneous polynomial:

$$f(x, y) = x^k + y^k, \quad k > 1,$$

then the radial vector field $v = (x, y)$ is obviously tangent to V, singular only at 0, and a straight-forward computation using the formulas in Sect. 2 below shows that its GSV index (which equals the homological one) is $1-(k-1)^2$, which is negative if $k > 2$. However the results in [16] show that in all cases there is a smallest possible index that the holomorphic vector fields on an isolated singularity germ $(V, 0)$ may attain, and this index is attained by an open and dense set in the space of germs of holomorphic vector fields at 0; what is this lower bound?

Obviously this provides an invariant of the singularity which is interesting to study. This is closely related to asking what is a generic vector field on a singular variety?

Conjecture: Let $(V, 0)$ be an ICIS germ of dimension $n \geq 1$, with Milnor number $\mu \geq 1$, and let v be a holomorphic vector field on V with an isolated singularity at 0. Then the Poincaré–Hopf index of any continuous extension of v to a nearby Milnor fiber of V is $\geq 1 + (-1)^n \mu$.

It is not hard to show that this conjecture is true for curves, *i.e.*, when V has complex dimension 1, and for quasi-homogeneous germs of all dimensions.

7.2 The Hypersurface Case

Let us restrict the discussion to the case when V is a hypersurface, following [68]. The aim now is twofold: on one hand we want to give an algebraic expression to compute the homological index using linear algebra, as in the case of vector fields in $\mathbb{C}^n$. On the other hand we use this algebraic formula, together with the stability property 7.1.4 of the index, to show that for hypersurfaces the homological and the GSV indices coincide. To establish the algebraic facts in [68] that lead to these statements is not an easy task and requires hard computations, so we only sketch here some of the main points. We refer to Gómez-Mont's article for the actual proofs and details. We refer to [17] for the equivalent arguments for the index of vector fields on complete intersection singularities.

Let $\mathbb{B} = \mathbb{B}^{n+1}$ be an open ball around the origin in $\mathbb{C}^{n+1}$ and let $f : \mathbb{B}^{n+1} \to \mathbb{C}$ be a holomorphic function with $0 \in \mathbb{C}^{n+1}$ as its only critical point. Hence the 1-form $\sum_{i=1}^{n+1} \frac{\partial f}{dz_i} dz_i$ vanishes only at $0 \in \mathbb{C}^{n+1}$, where $\frac{\partial f}{dz_i}$ denotes the partial derivative of f with respect to z_i. Let $\mathcal{I}_f$ be the Jacobian ideal $(\frac{\partial f}{dz_1}, \cdots, \frac{\partial f}{dz_{n+1}}) \subset \mathcal{O}_{\mathbb{B},0}$ of f. Given a vector field v tangent to V, with a unique singularity at 0, restriction of a vector field $\widehat{v} = (a_1, \cdots, a_{n+1})$ on $\mathbb{B}$ with a unique singularity at the origin, it is shown in [68, Theorem 2] that the homology groups $H_i(\Omega^\bullet_{V,0}, v)$ of the complex (7.1.2) have dimensions:

$$\begin{aligned} h_0(\Omega^\bullet_{V,0}, v) &= \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, a_1, \cdots, a_{n+1}), \\ h_n(\Omega^\bullet_{V,0}, v) &= \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, \mathcal{I}_f), \\ h_i(\Omega^\bullet_{V,0}, v) &= \lambda, \quad \text{for} \quad i = 1, \ldots, n-1, \end{aligned}$$

where $0 \leq \lambda \leq \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, \mathcal{I}_f)$ is the integer:

$$\begin{aligned} \lambda = &\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, a_1, \cdots, a_{n+1}) + \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(\frac{df}{f}(\widehat{v}), a_1, \cdots, a_{n+1}) \\ &- \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(a_1, \cdots, a_{n+1}), \end{aligned}$$

noticing that the tangency condition for $\widehat{v}$ means that $\widehat{v}(df)$ is a multiple of f, so that $\frac{df}{f}(\widehat{v})$ is a holomorphic function on $\mathbb{B}$. As a consequence of these computations Gómez-Mont deduced the following expressions for the homological index (Theorem 1 in [68]).

Theorem 7.2.1. *Let $(V, 0)$ be an isolated hypersurface singularity (germ) of dimension n in $\mathbb{C}^{n+1}$, and let v be the restriction to V of a holomorphic vector field $\widehat{v}$ on a neighborhood $\mathbb{B}$ of 0 in $\mathbb{C}^{n+1}$, which has an isolated singularity at 0 and is tangent to V.*

(1) *If n is odd, then the homological index of v is:*

$$\mathrm{Ind}_{\mathrm{hom}}(v, 0; V) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, a_1, \cdots, a_{n+1}) - \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, \mathcal{I}_f).$$

(2) *If n is even, then:*

$$\begin{aligned} \mathrm{Ind}_{\mathrm{hom}}(v, 0; V) = &\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(a_1, \cdots, a_{n+1}) \\ &- \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(\frac{df}{f}(\widehat{v}), a_1, \cdots, a_{n+1}) + \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, \mathcal{I}_f). \end{aligned}$$

The next step is easy once we have Theorem 7.1.3 above, *i.e.*, the stability of the index under perturbations. In fact, one has:

Lemma 7.2.1. *The difference*

$$\mathrm{Ind}_{\mathrm{hom}}(v,0;V) \; - \; \mathrm{Ind}_{\mathrm{GSV}}(v,0;V) \; = \; k,$$

is a constant that depends only on **K** *and not on the choice of vector field.*

The proof of this lemma is an exercise: both indices satisfy a "Law of conservation of Number"; in the case of the homological index this is Theorem 7.1.3, while for the GSV index this is an immediate consequence of the fact that it equals the Poincaré–Hopf index of any continuous extension of the vector field to the Milnor fiber. This implies that the difference between both indices (GSV and homological) is locally constant on the space $\Theta_0(V,0)$ of germs at 0 of holomorphic vector fields tangent to V, because both indices coincide with the usual index of Poincaré–Hopf on V_{reg}. Hence this difference is constant on all of $\Theta_0(V,0)$, since this space is connected. □

Notice that for the previous lemma we did not use the fact that V is a hypersurface: the same statement holds for local complete intersections, and in general for every isolated singularity germ if we replace the GSV index by the Schwartz index (of course in that case the constant k will be different).

Thus, in order to prove that for hypersurface singularities the GSV and homological indices coincide, we only need to find an appropriate vector field for which one can show that the constant k vanishes, and this is what Gómez-Mont does in the last Sect. 3.2 of [68]. It is proved in [17] that a similar argument works to identify both indices on complete intersection germs. The hard, and possibly most interesting, part in that case is to find algebraic formulae for the index that actually allow explicit computations.

To complete the arguments in the hypersurface case one must distinguish again between even and odd dimensions. If the dimension of V is odd, so that V is defined by the holomorphic map (germ) $(\mathbb{C}^{2n},0) \xrightarrow{f} (\mathbb{C},0)$, then one has the Hamiltonian vector field

$$\zeta = \Big(\frac{\partial f}{\partial z_2}, -\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{2n}}, -\frac{\partial f}{\partial z_{2n-1}}\Big)$$

defined on all of $\mathbb{C}^{2n}$ and with 0 as its only singularity. The differential of f is:

$$df(z_1,\cdots,z_{2n}) = \frac{\partial f}{\partial z_1}(z_1,\cdots,z_{2n})\,dz_1 + \cdots + \frac{\partial f}{\partial z_{2n}}(z_1,\cdots,z_{2n})\,dz_{2n}.$$

Thus one has:

$$df(\zeta) \; \equiv \; 0$$

everywhere on $\mathbb{C}^{2n} \setminus \{0\}$. This means that ζ is not only tangent to the hypersurface $V := f^{-1}(0)$, but it is also tangent to all the nonsingular hypersurfaces $V_t := f^{-1}(t)$, and with no singularity on each V_t, $t \neq 0$. This implies that its GSV index is 0. On the other hand, in this case Theorem 7.2.1 says that the homological index is:

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{2n},0}/(f, a_1, \cdots, a_{2n}) \; - \; \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{2n},0}/(f, \mathcal{I}_f) \; = \; 0,$$

since in this case the components of ζ are the partial derivatives of f, so they generate the Jacobian ideal $\mathcal{I}_f$. Hence both indices coincide for vector fields on hypersurface singularities of odd dimension.

When the dimension of V is even the considerations are similar in spirit: one constructs a holomorphic vector field on every such singularity germ, whose homological and GSV indices can be computed explicitly and to show that they coincide. However the construction of such examples is not that simple now. Let us denote the coordinates of $\mathbb{C}^{2n+1}$ by $(z_0, z_1, \cdots, z_{2n})$ and consider the vector field:

$$\widehat{v} \; = \; \Big(f\,, \frac{\partial f}{\partial z_2}\,, -\frac{\partial f}{\partial z_1}\,, \cdots, \frac{\partial f}{\partial z_{2n}}\,, -\frac{\partial f}{\partial z_{2n-1}}\Big),$$

We assume we have chosen the coordinates in such a way that $\widehat{v}$ has an isolated singularity at 0, *i.e.*, that the hypersurface $V = \{f = 0\}$ meets only at 0 the set $\bigcap_{i=1}^{2n} \{\frac{\partial f}{\partial z_i} = 0\}$. We set $v \; = \; \widehat{v}|_V$. Notice one has:

$$\frac{df}{f}(\widehat{v}) \; = \; \frac{\partial f}{\partial z_0},$$

hence v is tangent to V. Since $\widehat{v}$ has an isolated singularity at 0, the homological index of v on V can be computed using $\widehat{v}$ in the formula 7.2.1. One has:

$$\mathrm{Ind}_{\mathrm{hom}}(v, 0; V) \; = \; \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{B},0}/(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_{2n}}).$$

Now we must compute the GSV index of v and compare it with the homological index. For this Gómez-Mont notices that v is also the restriction to V of the holomorphic vector field:

$$\widehat{u} \; = \; \Big(0\,, \frac{\partial f}{\partial z_2}\,, -\frac{\partial f}{\partial z_1}\,, \cdots, \frac{\partial f}{\partial z_{2n}}\,, -\frac{\partial f}{\partial z_{2n-1}}\Big),$$

which is tangent to all the nonsingular hypersurfaces $f^{-1}(t)$, $t \neq 0$. The singular set of $\widehat{u}$ is the complete intersection curve defined by the ideal $(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_{2n}})$, which meets each nonsingular fiber $f^{-1}(t)$ at finitely many points, whose total sum (counting multiplicities) is the GSV-index of v on V. A direct computation then shows that this index equals the homological one. Thus we arrive to the following Theorem 3.5 of [68]:

Theorem 7.2.2. *Let* $(V, 0)$ *be an isolated hypersurface germ in* $\mathbb{C}^{n+1}, n>1$. *Then the homological index of every holomorphic vector field on* V *with an isolated singularity, equals the Poincaré–Hopf index of every continuous extension of* v *to a Milnor fiber of* V.

Remark 7.2.1. 1. The formula of Gómez-Mont in 7.2.1 requires that the extension $\widehat{v}$ of the vector field v to the ambient space also has an isolated singularity. O. Klehn in [97] gives an algebraic formula for the index that only requires $\widehat{v}$ to have an isolated singularity on V. In [98] the same author studies the particular case of vector fields that extend to the ambient space being tangent to the fibers of some smoothing and he gives an algebraic formula for the index of such holomorphic vector fields when the ambient space has dimension 1 (compare with the algebraic formula in [71]). This formula also applies to the real analytic case.

2. If V is a compact hypersurface with isolated singularities in a complex manifold M and one has a global holomorphic vector field on V with isolated singularities, then one can define its *total homological index* in the obvious way. Since this index coincides with the GSV-index, which coincides with the virtual index, it follows that the total homological index is independent of the choice of vector field and equals the 0-degree Fulton–Johnson class of V, *i.e.*, the Euler–Poincaré characteristic of a global smoothing of V (see Chap. 11). B. Khanedani in [92] extended the definition of the homological index to sections of general linear fibered spaces over complex analytic spaces, and proved that the total sum of indices is independent of the choice of section. It would be interesting to determine what this sum is.

3. The definition of the homological index was extended in [65] to holomorphic vector fields with an isolated singularity on hypersurface germs with nonisolated singularities; an algebraic formula to compute this index is also given in that article when the singular locus has dimension 1. We know from [38] that every complex analytic germ $(V, 0)$ has a *logarithmic* stratification which is Whitney regular when the germ is *holonomic* (*i.e.*, when the stratification has finitely many strata). In this case it is natural to ask if the homological index also coincides with the GSV-index defined in Chap. 3 if one considers the logarithmic stratification.

7.3 The Index of Real Analytic Vector Fields

It is natural to ask whether one can also find an algebraic formula for computing the index of a C^∞, or real analytic, vector field at an isolated singularity. It seems that this question was first raised by V. I. Arnold (c.f. [172]), and the answer was given independently by D. Eisenbud and H. Levine in [58] and by G.N. Khimshiashvili in [94], proving that the index can be computed as the signature of an appropriate bilinear form. This signature formula is proved in [10] by means of a limit process concerning functions on a finite set with involution, while the proof in [58] is of an algebraic nature and has a number of important consequences. This formula, usually known as *the Eisenbud–Levine formula*, was used by Arnold in [9] to give an upper bound for the index of homogeneous real analytic vector fields with an isolated singularity in terms of the degrees of the components. In [95] the signature formula is used to generalize the Petrovskii–Oleinik inequalities in real algebraic geometry. There is also an algebraic formula in [9, 10] for the index of gradient vector fields in terms of signatures of certain quadratic forms, and this leads to a formula for the Euler characteristic of the fibers of real valued analytic mappings. These results have also motivated interesting research in this vein for real analytic vector fields on singular varieties.

7.3.1 The Signature Formula of Eisenbud–Levine–Khimshiashvili

The formula of [58, 94] is somehow the paradigm of all the later algebraic formulas for indices of analytic vector fields on real analytic singular varieties.

Let us denote by $A_{\mathbb{R}^n,0}$ the local ring of germs of real analytic real-valued functions, and consider a germ of a vector field $v = (a_1, \cdots, a_n)$ at 0, where the components are elements in $A_{\mathbb{R}^n,0}$. We let B_v be the local algebra of v:

$$B_v := A_{\mathbb{R}^n,0}/(a_1, \cdots, a_n),$$

where $(a_1, \cdots, a_n)$ denotes the ideal generated by the components of v. The dimension of B_v as a vector space over $\mathbb{C}$ is the *multiplicity* $\lambda(v)$ of v at 0. If $\lambda(v) < \infty$, then v necessarily has an isolated zero at 0, but the converse is not necessarily true: $\lambda(v) < \infty$ is equivalent to saying that the complexification $v_{\mathbb{C}}$ of v has an isolated singularity at 0. In this case we say that the singularity of v at $0 \in \mathbb{R}^n$ is *algebraically isolated*. The signature formula of [58] and [94] deals with such vector fields. In fact the theorem in [58] is proved for C^∞ vector fields with an isolated singularity of "finite multiplicity."

It is noticed in [58] that the multiplicity alone does not determine the local index, as it does in the complex case, and one needs to get further information from the local ring in order to determine the index. For this, given an analytic vector field v with finite multiplicity, let J_v be the *Jacobian* of v, *i.e.*, the (local) function whose value at each point is the determinant of the matrix:

$$\begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \cdots & \frac{\partial a_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_n}{\partial x_1} & \cdots & \frac{\partial a_n}{\partial x_n} \end{pmatrix}$$

For simplicity we also denote by J_v the residue class of the Jacobian in the local ring B_v. Now observe that B_v is actually an algebra and given a linear functional $\phi : B_v \to \mathbb{R}$ one can define a map $B_v \times B_v \xrightarrow{\langle f,g \rangle} \mathbb{R}$ by:

$$\langle f, g \rangle \;=\; \langle f, g \rangle_\phi \;=\; \phi(fg),$$

so the map $\langle , \rangle$ is given by the composition,

$$B_v \times B_v \xrightarrow{\;\cdot\;} B_v \xrightarrow{\;\phi\;} \mathbb{R}.$$

This is clearly a bilinear form. Let $\mathrm{Sgn}(v)$ denote the signature of this bilinear form, *i.e.*, the number of positive eigenvalues minus the number of negative eigenvalues. Then one has the index formula of [58, 94]:

Theorem 7.3.1. *One can always choose the linear form ϕ so that $\phi(J_v) > 0$, and in this case one has:*

$$\mathrm{Ind}_{\mathrm{PH}}(v, 0) \;=\; \mathrm{Sgn}(v),$$

independently of the choice of ϕ.

We refer to either [58] or [10] for a proof of this theorem. As an example [58], consider the vector field on $\mathbb{R}^2 \cong \mathbb{C}$ given by $v(z) = z^2$, which we regard as the real analytic vector field $(x^2 - y^2, 2xy)$ with local index 2. A basis for its local algebra B_v is given by $\{1, x, y, J_v\}$, where $J_v \;=\; 4(x^2 + y^2\}$ is the Jacobian. One can take as ϕ the linear functional that takes J_v into 1 and all the other basis elements into 0. The matrix of the corresponding bilinear form is:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

whose signature is 2.

Remark 7.3.1. If the real analytic vector field has an isolated singularity that is not algebraically isolated, the signature formula for the index does not hold for several reasons. In [43] V. Castellanos used the Koszul complex analogous to (7.1.2) with real analytic forms replacing the holomorphic ones, to obtain a signature formula in this setting.

7.3.2 The Index on Real Hypersurface Singularities

We know that for real analytic hypersurfaces there is not "a" Milnor fiber but there are fibers to the left and to the right of a critical value, say $0 \in \mathbb{R}$, with possibly different topology. Therefore one can not define in general an index in the spirit of the GSV-index, *i.e.*, a well-defined integer associated to each vector field with an isolated singularity, which measures the number of zeroes of an extension of the vector field to a Milnor fiber: the number one gets depends on the choice of Milnor fiber (see [6] or Chap. 4 above).

If the hypersurface is odd dimensional, things are simpler from the topological viewpoint because the Euler–Poincaré characteristic of the Milnor fiber is well defined; however for even dimensions this is only well defined modulo 2. Still, the formulas by Arnold in [9, p. 3] show that for gradient vector fields, the algebra behind the function determines the Euler characteristic of the fibers in all cases: it is $1 \pm \sigma$ where σ is the signature of an appropriate bilinear form on the local ring (algebra) of the singularity.

This suggests that something can be done for vector fields in general on real analytic hypersurface singularities. This program was carried out independently (and differently) by Gómez-Mont and Mardešić on one hand ([69,70], see also [62,66]) and by Ebeling and Gusein-Zade [49] on the other hand.

The work in [49] is somehow inspired by the formulas in [78,166]; for this the authors define an index of vector fields with an isolated singularity, which is the *radial index* that we introduced in Chaps. 2 and 4, and then they give an algebraic formula to compute this index when the vector field is the gradient of a function.

The work of Gómez-Mont and Mardešić is closely related to the formula in [58,94]; let us have a glance of what they do. Let U be an open neighborhood around 0 in $\mathbb{R}^{n+1}$, and let $f : (U,0) \to (\mathbb{R},0)$ be analytic with an algebraically isolated singularity at 0 (*i.e.*, its complexification has an isolated singularity); set $V = f^{-1}(0)$. Let A be the local ring of f at 0 (an algebra in fact):

$$A \;=\; A_{\mathbb{R}^{n+1},0}/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n+1}}),$$

where $(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n+1}}) = J_f$ is the Jacobian ideal of f. This algebra is finite dimensional because of the assumption that V has an algebraically isolated singularity, and it has a distinguished element: the class of the Hessian,

$$\mathrm{Hess}(f) := \det\Big(\Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)\Big) \in A.$$

This class generates an ideal in A which is minimal in the sense that it is contained in every nonzero ideal of A (see [58,94]).

Now consider a real analytic vector field v on $V \cap U$, tangent to V and with an algebraically isolated singularity at 0. Thus v is the restriction to V of a real analytic vector field $\widehat{v}$ on a ball $\mathbb{B} \subset \mathbb{R}^{n+1}$, such that $df(v)(x) = 0$ for each $x \in V$. Since the ideal of functions vanishing on V is generated by f, one has that $df(v)$ is a multiple of f, so the assumption of v being tangent to V is actually equivalent to saying that there exists $h_v \in A_{\mathbb{R}^{n+1},0}$ so that $df(v) = fh_v$.

Following [69, 70], we consider the local algebra of $v = (a_1, \cdots, a_{n+1})$:

$$B_v \;=\; A_{\mathbb{R}^{n+1},0}/(a_1, \cdots, a_{n+1}).$$

This algebra is also finite dimensional because v has an algebraically isolated singularity, and it also has a distinguished element: the class of the Jacobian of v:

$$J_v := \det\left(\left(\frac{\partial a_i}{\partial x_j}\right)\right) \in B_v.$$

We know that in the signature formula of [58, 94] for the index (when the ambient space is smooth), the Jacobian J_v and the signature of a certain quadratic form determine the index. In the case envisaged here one must consider the *relative Jacobian* $J_f(v)$ and the *relative Hessian* Hess(f) introduced respectively in [69, 70]. It is shown that J_v is divisible by h_v in B_v and thus the relative Jacobian $J_f(v)$ is a well-defined element (see [69, p. 1528]),

$$J_f(v) := \frac{J_v}{h_v} \in B_v/\mathrm{Ann}_{B_v}(h_v),$$

where $\mathrm{Ann}_{B_v}(h_v)$ is the annihaltor, h_v being as above. It is proved that there is a linear map $\ell : B_v/\mathrm{Ann}(h) \to \mathbb{R}$ such that $\ell(J_f(v)) > 0$. The product in $B_v/\mathrm{Ann}(h)$ together with ℓ defines a bilinear form on $B_v/\mathrm{Ann}(h)$. Let $\mathrm{Sgn}_{V,0}(v)$ denote the signature of this bilinear form.

It is proved in [69] that the function $\mathrm{Sgn}_{V,0}$ "behaves like an index" in the sense that for n even it satisfies the law of conservation of number:

$$\mathrm{Sgn}_{V,0}(v) = \mathrm{Sgn}_{V,0}(v_t) + \sum_{\substack{x \in V\setminus\{0\} \\ v_t(x)=0}} \mathrm{Index}_{\mathrm{PH}}(v_t, x; V \setminus \{0\})$$

for x close to 0 and v_t tangent to V and close to v. The same formula holds for n odd under a certain additional hypothesis.

Similarly, the *relative Hessian* is defined in [70] by:

$$\mathrm{Hess}_{\mathrm{rel}}(f) := \frac{\mathrm{Hess}(f)}{h_v} \in A/\mathrm{Ann}_A(h_v).$$

It is shown in [70] that one can also construct a linear functional ℓ' on $A/\mathrm{Ann}_A(h_v)$ so that:

$$\ell'(\mathrm{Hess}_{rel}(f)) > 0;$$

the construction of this functional follows the line of the general theory developed in [10,58,94]. As before, one may use this functional to define a bilinear form. Let $\mathrm{Sgn}_A(h_v)$ denote the signature of this bilinear form.

Now, if V has odd dimension $n \geq 1$, the Euler–Poincaré characteristic of the fibers $V_t = f^{-1}(t) \cap \mathbb{D}_\varepsilon$, $t \neq 0$ is well defined, and so is the index $\mathrm{Ind}_{\mathrm{GSV}}$ defined in Chap. 3: it is the Poincaré–Hopf index of an extension of v to a non-singular fiber $f^{-1}(t)$, $t \neq 0$. In this case Theorem 1 of [70] gives an algebraic formula for the index of an analytic vector field which is tangent to V:

Theorem 7.3.2. *Let $n \geq 1$ be an odd integer, let $V = f^{-1}(0) \subset \mathbb{R}^{n+1}$ be a real analytic hypersurface with an algebraically isolated singularity at 0, and let v be a real analytic vector field on V with an algebraically isolated singularity at 0. Then:*

$$\mathrm{Ind}_{\mathrm{GSV}}(v) = \mathrm{Sgn}_{(V,0)}(v) - \mathrm{Sgn}_A(h_v),$$

where $h_v = df(v)/f \in A$.

We refer to [70] for the proof of this result and for several explanations giving insights of the geometry and algebra behind this formula. Notice that if V is regular at 0, then this formula reduces to the one in [58,94]. We refer to [69] for a discussion when V is even-dimensional.

Chapter 8
The Local Euler Obstruction

Abstract The local Euler obstruction was first introduced by R. MacPherson in [117] as an ingredient for the construction of characteristic classes of singular complex algebraic varieties. An equivalent definition was given by J.-P. Brasselet and M.-H. Schwartz in [33] using vector fields. Their viewpoint brings the local Euler obstruction into the framework of "indices of vector fields on singular varieties," though the definition only considers radial vector fields. This approach is most convenient for our study which is based on [29, 32] and shows relations with other indices. There are various other definitions and interpretations, in particular due to Gonzalez-Sprinberg [72], Verdier, Lê-Teissier and others. The survey [27] provides an overview on the subject.

Section 1 below is devoted to the definition of the local Euler obstruction and some of its main properties. The behavior of the local Euler obstruction relatively to hyperplane sections is described in Sect. 2, following [29]. In Sect. 3 and the thereafter we study a generalization of the local Euler obstruction introduced in [32] and called the Euler obstruction of the function, or also the "Euler defect"; this is an invariant associated to map-germs on singular varieties. MacPherson's local Euler obstruction corresponds to the square of the function distance to the given point. It is shown in [150], and explained in the last section of this chapter, that this invariant can be expressed in terms of the number of critical points in the regular part of a Morsification of the function.

8.1 Definition of the Euler Obstruction. The Nash Blow Up

We begin by recalling the definition of the Nash transformation of a singular variety V of dimension n. Since the definition is local we may restrict to germs of varieties.

Let $(V,0)$ be a reduced, pure-dimensional complex analytic singularity germ of dimension n in an open set $U \subset \mathbb{C}^m$. Let $G(n,m)$ denote the Grassmanian of complex n-planes in $\mathbb{C}^m$. On the regular part V_{reg} of V there

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is a map $\sigma : V_{\text{reg}} \to U \times G(n,m)$ defined by $\sigma(x) = (x, T_x(V_{\text{reg}}))$. The *Nash transformation* $\widetilde{V}$ of V is the closure of $\text{Im}(\sigma)$ in $U \times G(n,m)$. It is a (usually singular) complex analytic space endowed with an analytic projection map

$$\nu : \widetilde{V} \longrightarrow V$$

which is a biholomorphism away from $\nu^{-1}(\text{Sing}(V))$. Notice that each point $y \in \text{Sing}(V)$ is being replaced by all limits of planes $T_{x_i}V_{\text{reg}}$ for sequences $\{x_i\}$ in V_{reg} converging to y.

Let us consider the tautological bundle over $G(n,m)$ and denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(n,m)$. We denote by π the projection map of this bundle. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{V}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{V}$ is called *the Nash bundle* of V. An element of $\widetilde{T}$ is written (x, P, v) where $x \in U$, P is a n-plane in $\mathbb{C}^m$ based at x and v is a vector in P. So we have maps:

$$\widetilde{T} \xrightarrow{\pi} \widetilde{V} \xrightarrow{\nu} V.$$

Let us consider a complex analytic stratification $(V_\alpha)_{\alpha \in A}$ of V satisfying the Whitney conditions. Adding the stratum $U \setminus V$ we obtain a Whitney stratification of U. Let us denote by $TU|_V$ the restriction to V of the tangent bundle of U. We know that a stratified vector field v on V means a continuous section of $TU|_V$ such that if $x \in V_\alpha \cap V$ then $v(x) \in T_x(V_\alpha)$. By Whitney condition (a) one has the following lemma of [33]:

Lemma 8.1.1. *Every stratified vector field v on a subset $A \subset V$ has a canonical lifting to a section $\widetilde{v}$ of the Nash bundle $\widetilde{T}$ over $\nu^{-1}(A) \subset \widetilde{V}$.*

Now consider a stratified radial vector field $v(x)$ in a neighborhood of $\{0\}$ in V, *i.e.*, there is ε_0 such that for every $0 < \varepsilon \le \varepsilon_0$, $v(x)$ is pointing outwards the ball $\mathbb{B}_\varepsilon$ over the boundary $\mathbb{S}_\varepsilon := \partial \mathbb{B}_\varepsilon$.

The following interpretation of the local Euler obstruction has been given by Brasselet–Schwartz [33]. We refer to [117] for the original definition which uses 1-forms instead of vector fields (see also Chap. 9 below).

Definition 8.1.1. Let v be a radial vector field on $V \cap \mathbb{S}_\varepsilon$ and $\widetilde{v}$ the lifting of v on $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$ to a section of the Nash bundle. The *local Euler obstruction* (or simply the Euler obstruction) $\text{Eu}_V(0)$ is defined to be the obstruction to extending $\widetilde{v}$ as a nowhere zero section of $\widetilde{T}$ over $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$.

More precisely, let $\mathcal{O}(\widetilde{v}) \in H^{2d}\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon)\big)$ be the obstruction cocycle to extending $\widetilde{v}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$. The local Euler obstruction $\text{Eu}_V(0)$ is defined as the evaluation of the cocycle $\mathcal{O}(\widetilde{v})$ on the fundamental class of the pair $\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon)\big)$. The Euler obstruction is an integer.

Remark 8.1.1. We notice that for $\varepsilon > 0$ small enough, if $\widetilde{v}$ is a nowhere zero section of $\widetilde{T}$ defined on $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$ which lifts a vector field transverse to $V \cap \mathbb{S}_\varepsilon$, then the Euler obstruction $\mathrm{Eu}_V(0)$ equals the obstruction to extending $\widetilde{v}$ as a nowhere zero section of $\widetilde{T}$ over $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$. This is a consequence of the fact that every section $\widetilde{v}$ as above is homotopic to a section of $\widetilde{T}$ over $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$ obtained by lifting a radial vector field of V at 0. Hence, to calculate the Euler obstruction of $(V, 0)$ it suffices to construct a nowhere zero section of $\widetilde{T}$ defined on $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$ which lifts a vector field transverse to $V \cap \mathbb{S}_\varepsilon$. Of course one also needs to understand how this section extends to $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$.

The following result summarizes some basic properties of the Euler obstruction:

Theorem 8.1.1. *The Euler obstruction satisfies:*

(1) $\mathrm{Eu}_V(x) = 1$ *if x is a regular point of V.*

(2) $\mathrm{Eu}_{V \times V'}(x \times x') = \mathrm{Eu}_V(x) \cdot \mathrm{Eu}_{V'}(x')$.

(3) *If V is locally reducible at x and V_i are its irreducible components, then* $\mathrm{Eu}_V(x) = \sum \mathrm{Eu}_{V_i}(x)$.

(4) $\mathrm{Eu}_V(x)$ *is a constructible function on V, in fact it is constant along the strata of a Whitney stratification.*

These statements are all contained in [117], except for (iv) which is implicitly stated there and we refer to [33] for a detailed proof. Now we have the following result of [33].

8.1.1 Proportionality Theorem for Vector Fields

In this section, we prove the Proportionality Theorem for vector fields in [33]:

Theorem 8.1.2. *Let v be a stratified vector field on V which is obtained by radial extension in a neighborhood of a singularity $x \in V_\alpha$. Let $\widetilde{v}$ be the canonical lifting of v to a section of the Nash bundle $\widetilde{T}$ over the boundary of $\nu^{-1}(V \cap \mathbb{B}_\varepsilon(x))$, where $\mathbb{B}_\varepsilon(x)$ is a small ball around x in $\mathbb{C}^m$. Let $\mathcal{O}(\widetilde{v}) \in H^{2n}\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon(x)), \nu^{-1}(V \cap \mathbb{S}_\varepsilon(x))\big)$ be the obstruction cocycle to extending $\widetilde{v}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(V \cap \mathbb{B}_\varepsilon(x))$ and let $\mathrm{Eu}_V(v, x)$ be the evaluation of $\mathcal{O}(\widetilde{v})$ on the fundamental class of the pair $\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon(x)), \nu^{-1}(V \cap \mathbb{S}_\varepsilon(x))\big)$. Then one has:*

$$\mathrm{Eu}_V(v, x) = \mathrm{Ind}_{\mathrm{PH}}(v, x; V_\alpha) \cdot \mathrm{Eu}_V(x) \tag{8.1.2}$$

where $\mathrm{Ind}_{\mathrm{PH}}(v, x; V_\alpha)$ is the Poincaré–Hopf index at x of the restriction of v to the stratum that contains x.

In short this theorem says that the obstruction $\mathrm{Eu}_V(v, x)$ to extend the lifting $\widetilde{v}$ as a section of the Nash bundle inside $\nu^{-1}(V \cap \mathbb{B}_\varepsilon(x))$ is proportional to the Poincaré–Hopf index of v at x, the proportionality factor being precisely the local Euler obstruction.

Let p be a point in a stratum V_α with $n_\alpha = \dim_{\mathbb{C}} V_\alpha > 0$. Let $\mathbb{B}$ a ball around p in M, small enough so that TV_α is trivial on $\mathbb{B} \cap V_\alpha$, and set $\mathbb{S} = \partial\mathbb{B}$. We denote by $T^\times M$, $T^\times V_\alpha$ and $\widetilde{T}^\times$ the bundles obtained from TM, TV_α and $\widetilde{T}$, respectively, by removing the zero sections.

We denote by $\theta(M, V; p)$ the set of homotopy classes of stratified vector fields on V obtained by radial extension of some vector field around p on V_α. We remark that, by [48] Theorem 1.1, this set coincides with the set of usual homotopy classes of stratified vector fields.

On the other hand, we denote by $\theta(V_\alpha, p)$ the set of (usual) homotopy classes of vector fields of V_α, defined and nonvanishing on $\mathbb{S} \cap V_\alpha$. Note that $\mathbb{S}_\alpha := \mathbb{S} \cap V_\alpha$ is a $(2n_\alpha - 1)$-sphere, so that such a vector field v_α defines a map

$$\varphi_{v_\alpha} : \mathbb{S}_\alpha \xrightarrow{v_\alpha} T^\times V_\alpha|_{\mathbb{S}_\alpha} \xrightarrow{h} \mathbb{S}_\alpha \times \mathbb{C}^{n_\alpha} \setminus \{0\} \xrightarrow{pr_2} \mathbb{C}^{n_\alpha} \setminus \{0\},$$

where h is an isomorphism.

This correspondence induces a bijection of $\theta(V_\alpha, p)$ onto the homotopy group $\pi_{2n_\alpha - 1}(\mathbb{C}^{n_\alpha} \setminus \{0\}) \simeq \mathbb{Z}$, where the isomorphism is given by the mapping degree. A vector field v_α as above may be extended to a vector field on $\mathbb{B}$ with isolated singularity at p. Recall that, by definition, $\mathrm{Ind}_{\mathrm{PH}}(v_\alpha, V_\alpha; p)$ is the mapping degree of φ_{v_α}. In summary, $\theta(V_\alpha, p) \simeq \mathbb{Z}$, which is generated by the class $[v_{\alpha,rad}]$ of a radial vector field on V_α, singular at p.

Lemma 8.1.2. *There is a natural bijection between $\theta(M, V; p)$ and $\theta(V_\alpha, p)$. Thus*

$$\theta(M, V; p) \simeq \mathbb{Z}.$$

It is generated by the class $[v_{rad}]$ of a stratified radial vector field. Moreover, the elements in $\theta(M, V; p)$ are classified by their local Schwartz index at p.

Proof. Note that the map

$$\theta(M, V; p) \longrightarrow \theta(V_\alpha, p)$$

given by restriction is well-defined. It is surjective by the radial extension process above. We also see that it is injective by applying a similar construction to homotopies on $\mathbb{S}_\alpha$.

Let us recall the classical construction to describe the element $[k \cdot v_{\alpha,rad}]$ in $\theta(V_\alpha, p)$. If $k > 0$, let $\vee^k \mathbb{S}_\alpha$ denote the bouquet of k copies of $\mathbb{S}_\alpha$ obtained by collapsing to a point k half-spheres of dimension $(2n_\alpha - 2)$ in $\mathbb{S}_\alpha$ through the north and south poles. We have a collapsing map $\kappa : \mathbb{S}_\alpha \to \vee^k \mathbb{S}_\alpha$ and a map $\varphi : \vee^k \mathbb{S}_\alpha \to \mathbb{C}^{n_\alpha} \setminus \{0\}$, which is equal to $\varphi_{v_{\alpha,rad}}$ on each $\mathbb{S}_\alpha$. Then, we define the map $k \cdot \varphi_{v_{\alpha,rad}}$ as the composition

$$\varphi \circ \kappa : \mathbb{S}_\alpha \longrightarrow \mathbb{C}^{n_\alpha} \setminus \{0\}.$$

The element in $\theta(V_\alpha, p)$ corresponding to $[k \cdot v_{\alpha,rad}]$ is the homotopy class of the vector field $k \cdot v_{\alpha,rad} := h^{-1}(x, \varphi \circ \kappa(x))$ on $\mathbb{S}_\alpha$. If $k = -\lambda < 0$, one can provide a similar description taking the bouquet of λ spheres and using, instead of $v_{\alpha,rad}$, a real linear diagonal vector field in $\mathbb{C}^{n_\alpha}$ having the $2n_\alpha$-vector $(-1, 1, \cdots, 1)$ as its diagonal coefficients.

Proof of Theorem 8.1.2. Let $n_\alpha = \dim_{\mathbb{C}} V_\alpha$. If $n_\alpha = 0$, then $v = v_{\text{rad}}$ and we have identity 8.1.2. Thus we assume that $n_\alpha > 0$ hereafter.

We denote by v_{rad} a stratified radial vector field at p. The vector field v is stratified and nonvanishing on a small sphere $\mathbb{S}$ around p. By Lemma 8.1.2, we have a stratified homotopy

$$\psi : (\mathbb{S} \cap V) \times [0,1] \longrightarrow T^\times M|_{\mathbb{S}\cap V}$$

between v and $k \cdot v_{\text{rad}}$, $k = \text{Ind}_{\text{Sch}}(v, V; p)$. Here $k \cdot v_{\text{rad}}$ denotes the radial extension of the vector field $k \cdot v_{\alpha,rad}$ on $\mathbb{S} \cap V$. Thus we have

$$\partial \operatorname{Im} \psi = v(\mathbb{S} \cap V) - k \cdot v_{\text{rad}}(\mathbb{S} \cap V)$$

as chains in $T^\times M|_{\mathbb{S}\cap V}$. Since ψ is stratified, we can lift it to a homotopy

$$\widetilde{\psi} : \nu^{-1}(\mathbb{S} \cap V) \times [0,1] \longrightarrow \widetilde{T}^\times|_{\nu^{-1}(\mathbb{S}\cap V)}$$

and we have

$$\partial \operatorname{Im} \widetilde{\psi} = \widetilde{v}(\nu^{-1}(\mathbb{S} \cap V)) - (\widetilde{k \cdot v_{\text{rad}}})(\nu^{-1}(\mathbb{S} \cap V))$$

as chains in $\widetilde{T}^\times|_{\nu^{-1}(\mathbb{S}\cap V)}$.

The description following Lemma 8.1.2 shows that

$$(\widetilde{k \cdot v_{\text{rad}}})(\nu^{-1}(\mathbb{S} \cap V)) = k \cdot \widetilde{v_{\text{rad}}}(\nu^{-1}(\mathbb{S} \cap V))$$

as chains in $\widetilde{T}^\times|_{\nu^{-1}(\mathbb{S}\cap V)}$.

Taking a triangulation or a cellular decomposition of $\nu^{-1}(\mathbb{B} \cap V)$ and extending the homotopy $\widetilde{\psi}$ to the $(2n-1)$-skeleton of the decomposition, we see that the obstruction to extending $\tilde{v}$ is $k = \text{Ind}_{\text{Sch}}(v, V; p)$ times the obstruction to extending $\widetilde{v}_{\text{rad}}$. By definition of the Euler obstruction, we have the theorem. □

8.2 Euler Obstruction and Hyperplane Sections

The idea of studying the Euler obstruction "à la" Lefschetz, using hyperplane sections, is found in the work of Dubson [46] and Kato [88]. Also in [106] there are results in this spirit for the Euler obstruction and also for the Chern classes of singular varieties. The approach we follow here is that of [29, 32], which is topological.

We start with the following lemma, which is a special case of well-known results about Lefschetz pencils. Let us denote by $\mathcal{L}$ the space of complex linear forms on $\mathbb{C}^m$. Fix a Whitney stratification of V. There are a finite number of strata of this Whitney stratification which contain 0 in their closure, and we assume that the representative of $(V,0)$ is chosen small enough so that these are the only strata of V.

Lemma 8.2.1. *[29] There exists a nonempty Zariski open set Ω in $\mathcal{L}$ such that for every $l \in \Omega$, there exists a representative V of $(V,0)$ so that:*

(1) *for each $x \in V$, the hyperplane $l^{-1}(0)$ is transverse in $\mathbb{C}^m$ to every limit of tangent spaces in $TV_{\rm reg}$ of points in $V_{\rm reg}$ converging to x,*

(2) *for each y in the closure $\overline{V}_\alpha$ in V of each strata V_α, $\alpha = 1,\dots,\ell$, the hyperplane $l^{-1}(0)$ is transverse in $\mathbb{C}^m$ to every limit of tangent spaces in TV_α of points converging to y.*

In particular, for each $l \in \Omega$ one has for the Nash transformation

$$\widetilde{V} \subset \mathbb{C}^m \times (G(n,m) \setminus H^*),$$

where $H^* := \{T \in G(n,m)$ such that $l(T) = 0\}$.

Then we can state the following Theorem:

Theorem 8.2.1. *[29] Let $(V,0)$ be a germ of an equidimensional complex analytic space in $\mathbb{C}^m$. Let V_α, $\alpha = 1,\dots,\ell$, be the (connected) strata of a Whitney stratification of a small representative V of $(V,0)$ such that 0 is in the closure of every stratum. Then for each $l \in \Omega$ as in 8.2.1 there is ε_0 such that for any ε, $\varepsilon_0 > \varepsilon > 0$ and $t_0 \neq 0$ sufficiently small, we have the following formula for the Euler obstruction of $(V,0)$:*

$$\mathrm{Eu}_V(0) = \sum_{\alpha=1}^{\ell} \chi(V_\alpha \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)) \cdot \mathrm{Eu}_V(V_\alpha),$$

where χ denotes the Euler–Poincaré characteristic and $\mathrm{Eu}_V(V_\alpha)$ is the value of the Euler obstruction of V at any point of V_α, $\alpha = 1,\dots,\ell$.

Theorem 8.2.1 has been proved in [29], an alternative proof is given by Schürmann in [137]. We notice that the formula above is somehow in the spirit of the formula by Lê–Teissier in [106].

Remark 8.2.1. As noticed in [32], Theorem 8.2.1 can be stated through the framework of bivariant theory [26, 61] : the local Euler obstruction, as a constructible function, satisfies the local Euler condition with respect to general linear forms.

In the statement of Theorem 8.2.1 the closed ball may be replaced by an open ball, and/or the Euler characteristic may be replaced by the compactly supported Euler characteristic (see [26, 135]); these all agree locally. Also notice that the formula in 8.2.1 is similar to the index formula in [47].

Let us give some consequences of the theorem. We notice that the generic slice $V \cap \mathbb{B}_\varepsilon \cap l^{-1}(t_0)$ in 8.2.1 is by definition (see [73]) the *complex link* of 0 in V. In the case of an isolated singularity the complex link is smooth and there is only one stratum appearing in the sum in Theorem 8.2.1. In this case the theorem gives:

Corollary 8.2.1. *Let V be an equidimensional complex analytic subspace of $\mathbb{C}^m$ with an isolated singularity at 0. The Euler obstruction of V at 0 equals the GSV index of the radial vector field on a general hyperplane section $V \cap H$.*

The corollary is an immediate consequence of the theorem above and the definition of the GSV index in Chap. 3. We notice that this proves that the corresponding GSV index does not depend on the choice of the linear form.

In the case of a complete intersection of dimension n with isolated singularity the corresponding complex link is the Milnor fiber $\mathbf{F}$ of the linear function l. By [79] $\mathbf{F}$ has the homotopy type of a bouquet of spheres of real dimension $n-1$ and the number of such spheres is called the Milnor number μ of the singularity. Thus the Euler characteristic $\chi(\mathbf{F})$ equals $1 + (-1)^{n-1}\mu(V \cap H)$ and we have the formula of [46, 88], see also [106, (6.2.1)]:

Corollary 8.2.2. *Let V be a complex analytic complete intersection in $\mathbb{C}^m$ with an isolated singularity at 0. The Euler obstruction of V at 0 equals $1 + (-1)^{n-1}\mu^{(n-1)}$, where $\mu^{(n-1)}$ is the Milnor number at 0 of a general hyperplane section of V.*

Corollary 8.2.3. *Let V be an equidimensional complex analytic space of dimension n in $\mathbb{C}^m$ whose singular set $\mathrm{Sing}(V)$ is 1-dimensional at 0. Let l be a general linear form defined on $\mathbb{C}^m$ and denote by $\mathbf{F}_t$, the local Milnor fiber at 0 of the restriction of l to V. The singularities of $\mathbf{F}_t$ are the points $\mathbf{F}_t \cap \mathrm{Sing}(V) =: \{x_1, \ldots, x_m\}$. Then,*

$$\mathrm{Eu}_V(0) = \chi(\mathbf{F}_t) - m + \sum_1^m \mathrm{Eu}_V(x_i).$$

Proof. This is a consequence of 8.2.1, of the remark

$$\chi(\mathbf{F}_t) - m = \chi(\mathbf{F}_t - \mathrm{Sing}(V))$$

and of the fact that the local Euler obstruction at a nonsingular point is equal to 1. Observe that, since each x_i is an isolated singular point of $\mathbf{F}_t$, we can apply 8.2.1 to compute the right hand side of 8.2.3, because

$$\mathrm{Eu}_V(x_i) = \mathrm{Eu}_{\mathbf{F}_t}(x_i)\,.$$

We may also apply 8.2.2 if the singularities of $\mathbf{F}_t$ are complete intersections.

8.3 The Local Euler Obstruction of a Function

In this section we define an invariant introduced by J.P. Brasselet, D. Massey, A. J. Parameswaran and J. Seade in [32], which measures in a way how far the equality given in Theorem 8.2.1 is from being true if we replace the generic linear form l by some other function on V with at most an isolated stratified critical point at 0. For this it is convenient to think of the local Euler obstruction as defining an index for stratified vector fields. To be precise, let $(V,0)$ be again a complex analytic germ contained in an open subset U of $\mathbb{C}^m$ and endowed with a complex analytic Whitney stratification $\{V_\alpha\}$. We assume further that every stratum contains 0 in its closure. For every point $x \in V$, we will denote by $V_\alpha(x)$ the stratum containing x. We recall first some well-known concepts about singularity theory which originate in the work of R. Thom.

Let $f : V \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $\widehat{f} : U \to \mathbb{C}$. We recall [73] that a *critical point* of f is a point $x \in V$ such that $d\widehat{f}(x)(T_x(V_\alpha(x))) = 0$. We say, following [103], [73], that f has an isolated singularity at $0 \in V$ relative to the given Whitney stratification, if f has no critical points in a punctured neighborhood of 0 in V.

Let us denote by $\overline{\text{grad}}\widehat{f}(x)$ the conjugated gradient vector field of $\widehat{f}$ at a point $x \in U$, defined by $\overline{\text{grad}}\widehat{f}(x) := (\overline{\frac{\partial \widehat{f}}{\partial x_1}}, \ldots, \overline{\frac{\partial \widehat{f}}{\partial x_m}})$, where the bar denotes complex conjugation. From now on we assume that f has an isolated singularity at $0 \in V$. This implies that the kernel $\text{Ker}(d\widehat{f})$ is transverse to $T_x(V_\alpha(x))$ in any point $x \in V \setminus \{0\}$. Therefore at each point $x \in V \setminus \{0\}$, we have:

$$\text{Angle}\langle \overline{\text{grad}}\widehat{f}(x), T_x(V_\alpha(x))\rangle < \pi/2\,,$$

so the projection of $\overline{\text{grad}}\widehat{f}(x)$ on $T_x(V_\alpha(x))$, denoted by $\zeta_\alpha(x)$, is not zero.

Let V_β be a stratum such that $V_\alpha \subset \overline{V}_\beta$, and let $\pi : \mathcal{U}_\alpha \to V_\alpha$ be a tubular neighborhood of V_α in U. Following the construction of M.-H. Schwartz in [141, §2] we see that the Whitney condition (a) implies that at each point $y \in V_\beta \cap \mathcal{U}_\alpha$, the angle of $\zeta_\beta(y)$ and of the parallel extension of $\zeta_\alpha(\pi(y))$ is small. This property implies that these two vector fields are homotopic on the boundary of $\mathcal{U}_\alpha$. Therefore, we can glue together the vector fields ζ_α to obtain a stratified vector field on V, denoted by $\overline{\text{grad}}_V f$. This vector field is homotopic to $\overline{\text{grad}}\widehat{f}|_V$ and one has $\overline{\text{grad}}_V f \neq 0$ unless $x = 0$.

Definition 8.3.1. Let $\nu : \widetilde{V} \to V$ be the Nash transformation of V. We define *the local Euler obstruction of f on V* at 0, denoted $\text{Eu}_{f,V}(0)$, to be the Euler obstruction $\text{Eu}_{\overline{\text{grad}}_V f,V}(0)$ (see Definition 8.1.1) of the stratified vector field $\overline{\text{grad}}_V f$ at $0 \in V$.

In other words, let $\tilde{\zeta}$ be the lifting of $\overline{\text{grad}}_V f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{V}$ without singularity over $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$, where $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$ is the boundary of a small sphere around 0. Let $\mathcal{O}(\tilde{\zeta}) \in H^{2n}\big(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon)\big)$ be the obstruction cocycle to the extension of $\widetilde{\zeta}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$. Then the local Euler obstruction $\mathrm{Eu}_{f,V}(0)$ is the evaluation of $\mathcal{O}(\tilde{\zeta})$ on the fundamental class of the pair $(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon))$.

We notice that all these definitions and constructions also work when f is the restriction to V of a real analytic function on the ambient space. For instance, we can take f to be the function "distance to 0 on V," then $\overline{\mathrm{grad}}_V f$ is a radial vector field and the invariant $\mathrm{Eu}_{f,V}(0)$ is the usual local Euler Obstruction of V at 0.

We remark that the usual Hermitian metric on $\mathbb{C}^m$ defines a Riemannian metric which allows us to identify the real vector bundles $T\mathbb{C}^m$ and $T^*\mathbb{C}^m$. The latter is the holomorphic cotangent bundle and under the above identification $d\widetilde{f}$ corresponds to the conjugate gradient vector field.

A reason for considering the conjugate gradient vector field $\overline{\mathrm{grad}}_V f$, and not the usual gradient vector field $\mathrm{grad}_V f$, is given by the following Lemma, where f can be taken to be the restriction to V of either a real or complex analytic function on the ambient space. This lemma is also used in the proof of the main result in this chapter, Theorem 8.4.1.

Lemma 8.3.1. *Up to homotopy, the vector field $\overline{\mathrm{grad}}\widetilde{f}$ is the lifting of a constant vector field on $\mathbb{C}$, via $d\widetilde{f}$.*

Proof. The gradient vector field satisfies

$$d\widetilde{f}(\overline{\mathrm{grad}}\widetilde{f}(x)) = ||\overline{\mathrm{grad}}\widetilde{f}(x)||^2 \in \mathbb{R} \setminus \{0\} \qquad \text{for } x \in V \setminus \{0\},$$

so it is the lifting, up to scaling, of a constant vector field on a small disk in $\mathbb{C}$.

8.4 The Euler Obstruction and the Euler Defect

Now we have the following result of [32]; this compares the Euler obstruction of the space V with that of a function on V. According to Proposition 8.5.1, Theorem 8.2.1 is a special case of Theorem 8.4.1 taking f to be a general linear form.

Theorem 8.4.1. *Let $f : (V, 0) \to (\mathbb{C}, 0)$ have an isolated singularity at $0 \in V$. One has:*

$$\mathrm{Eu}_V(0) = \left(\sum_\alpha \chi(V_\alpha \cap \mathbb{B}_\varepsilon \cap f^{-1}(t_0)) \cdot \mathrm{Eu}_V(V_\alpha) \right) + \mathrm{Eu}_{f,V}(0).$$

In other words, the invariant $\mathrm{Eu}_{f,V}(0)$ measures the difference:

$$\mathrm{Eu}_V(0) - \left(\sum_\alpha \chi(V_\alpha \cap \mathbb{B}_\varepsilon \cap f^{-1}(t_0)) \cdot \mathrm{Eu}_V(V_\alpha) \right),$$

so it can be regarded as the "defect" for the local Euler obstruction of V to satisfy the Euler condition with respect to the function f. In this way one can generalize the definition of the Euler obstruction to functions with nonisolated singularities and one gets the *Euler defect* introduced in [32]. This arises as a natural application of Massey's work [118, 119] on intersections of characteristic cycles and derived categories.

The main step for proving 8.4.1 is the lemma below. To state this lemma we need some notation. We choose $\varepsilon > 0$ sufficiently small so that every sphere $\mathbb{S}_{\varepsilon'}$ in U centered at 0 and radius $\varepsilon' \leq \varepsilon$ intersects transversally every stratum in $V \setminus \{0\}$. Choose $\delta > 0$ small enough so that for each t in the disk $\mathbb{D}_\delta$ of radius δ around $0 \in \mathbb{C}$, the hypersurface $f^{-1}(t)$ intersects transversally the sphere $\mathbb{S}_\varepsilon$. Now choose ε' with $0 < \varepsilon' < \varepsilon$, and a point $t_0 \in \mathbb{D}_\delta$ such that $Y_{t_0} := f^{-1}(t_0)$ does not meet the sphere $\mathbb{S}_{\varepsilon'}$. We notice that the strata V_α intersect Y_{t_0} transversally and provide a Whitney stratification of this space.

Lemma 8.4.1. *There is a stratified vector field w on $V_{\varepsilon',\varepsilon} = V \cap (\mathbb{B}_\varepsilon \setminus \mathrm{Int}(\mathbb{B}_{\varepsilon'}))$ such that:*

(1) *it coincides with $\overline{\mathrm{grad}}_V f$ on $V \cap \mathbb{S}_{\varepsilon'}$ and its restriction to $V \cap \mathbb{S}_\varepsilon$ is radial; it is tangent to Y_{t_0},*

(2) *w has only a finite number of zeroes, and they are all contained in Y_{t_0},*

(3) *at each zero x, w is transversally radial to the stratum containing x (i.e., it is transverse to the boundary of a tubular neighborhood of the stratum).*

For the proof of Lemma 8.4.1, we refer the reader to [32]. The first steps of the proof are an interesting application of M.H. Schwartz techniques in order to construct the vector field w on a tube $\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta) \setminus \{0\}$, transverse (outwards pointing) to the boundary of the tube. The final step is to extend this vector field to all of V using Theorem 2.3 in [29]. Let us show how one deduces Theorem 8.4.1 from Lemma 8.4.1 (see [32]):

Proof of Theorem 8.4.1: (Assuming Lemma 8.4.1.) We first notice that if ξ is a stratified vector field on a neighborhood of $\{0\}$ in V which is everywhere transverse to a small sphere $\mathbb{S}_\varepsilon$, then ξ is homotopic to a radial vector field by elementary obstruction theory. Hence to compute the Euler obstruction it is enough to consider vector fields transverse to $\mathbb{S}_\varepsilon$.

The restriction of the vector field w of 8.4.1 to $\partial(V_{\varepsilon',\varepsilon})$ is a stratified vector field, so it can be lifted as a section $\widetilde{w}$ of the Nash bundle $\widetilde{T}$ on $\nu^{-1}(\partial(V_{\varepsilon',\varepsilon}))$ by 8.1.1. Let us denote by $\mathrm{Obs}(\widetilde{w}, \nu^{-1}(V_\varepsilon))$ the obstruction to extending $\widetilde{w}$ to $\nu^{-1}(V_\varepsilon)$. One has:

$$\mathrm{Obs}(\widetilde{w}, \nu^{-1}(V_\varepsilon)) = \mathrm{Obs}(\widetilde{w}, \nu^{-1}(V_{\varepsilon'})) + \mathrm{Obs}(\widetilde{w}, \nu^{-1}(V_{\varepsilon',\varepsilon}))$$

By statement (1) in Lemma 8.4.1 this formula becomes

$$\mathrm{Eu}_V(0) = \mathrm{Eu}_{f,V}(0) + \mathrm{Obs}(\tilde{w}, \nu^{-1}(V_{\varepsilon',\varepsilon}))\,.$$

By statement (3) in the Lemma 8.4.1 the contribution of $\mathrm{Obs}(\tilde{w}, \nu^{-1}(V_{\varepsilon',\varepsilon}))$ is concentrated on $\nu^{-1}(Y_{t_0} \cap \mathbb{B}_\varepsilon)$. Statements (3) and (4), together with the "Theorem of Proportionality" ([33], Théorème 11.1), Theorem 8.1.2 above, imply that the contribution of each singularity x of w to $\mathrm{Obs}(\tilde{w}, \nu^{-1}(V_{\varepsilon',\varepsilon}))$ is $\mathrm{Eu}_V(x)$-times the local Poincaré–Hopf index of w at x, regarded as a vector field on the stratum $V_\alpha(x)$. Furthermore (by (2) and (4)), the sum of the Poincaré–Hopf indices of the restriction of w to $V_\alpha \cap Y_{t_0}$ is $\chi(V_\alpha \cap Y_{t_0} \cap \mathbb{B}_\varepsilon)$, and Theorem 8.4.1 follows. □

Remark: The results of this section have been generalized in [76] to the case of functions with values in $\mathbb{C}^k$, $k \geq 1$.

8.5 The Euler Defect at General Points

By definition, if 0 is a smooth point of V and a regular point of f then $\mathrm{Eu}_{f,V}(0) = 0$ since in this case $\mathrm{Eu}_{f,V}(0)$ is the Poincaré–Hopf index of a vector field at a nonsingular point. In Proposition 8.5.1 below we prove that this is the case in a more general situation.

Definition 8.5.1. Let $(V, 0) \subset (U, 0)$ be a germ of analytic set in $\mathbb{C}^m$ equipped with a Whitney stratification and let $f : (V, 0) \to (\mathbb{C}, 0)$ be a holomorphic function, restriction of a regular holomorphic function $\widehat{f} : (U, 0) \to (\mathbb{C}, 0)$. We say that 0 *is a general point of* f if the hyperplane $\mathrm{Ker}\, d\widehat{f}(0)$ is transverse in $\mathbb{C}^m$ to every generalized tangent space at 0, *i.e.* to every limit of tangent spaces $T_{x_i}(V_\alpha)$, for every V_α and every sequence $x_i \in V_\alpha$ converging to 0.

We notice that for every f as above the general points of f form a nonempty open set on each (open) stratum of V, essentially by Sard's theorem. We also remark that this definition provides a coordinate free way of looking at the general linear forms considered in Theorem 8.2.1. In fact the previous definition is equivalent to saying that with an appropriate local change of coordinates $\widehat{f}$ is a linear form in U, and it is general with respect to V.

Proposition 8.5.1. *Let* 0 *be a general point of* $f : (V, 0) \to (\mathbb{C}, 0)$. *Then*

$$\mathrm{Eu}_{f,V}(0) = 0.$$

The proof of this Proposition is implicit within the proof of 2.3 in [29] and is also a consequence of Theorems 8.2.1 and 8.4.1 together. However we prove it here, following [32], for completeness and because this is how one deduces formula 8.2.1 from 8.4.1.

Proof. In a first step define the map

$$\widetilde{T} \subset (U \times G(n,m)) \times \mathbb{C}^m \xrightarrow{\widetilde{F}} \mathbb{D}_\delta \subset \mathbb{C}$$

by $\widetilde{F}(x,T,y) = d\widehat{f}_x(y)$, where $\mathbb{D}_\delta$ is a small disk around 0. As 0 is a general point of f, then $\widetilde{K} = \widetilde{T} \cap \widetilde{F}^{-1}(0)$ is a sub-bundle of $\tilde{T}$ of (complex) codimension 1 and $d\widetilde{F}$ maps the orthogonal complement of $\widetilde{K}$ isomorphically over $T(\mathbb{D}_\delta)$. In fact, since $\widehat{f}$ has an isolated singularity at 0 in V, away from 0 the kernel $\mathrm{Ker}(d\widehat{f})$ is transverse to each stratum and its orthogonal complement (in each stratum) determines a sub-bundle Q of $T\mathbb{C}^m|_{V\setminus\{0\}}$; the restriction of $d\widehat{f}$ to Q is an isomorphism between Q and $T(\mathbb{D}_\delta)$. Furthermore, since $\mathrm{Ker}(d\widehat{f})$ is transverse to each limit of tangent spaces at points $(x_i) \in V_{\mathrm{reg}}$ converging to 0, it follows that Q lifts to a sub-bundle of $\widetilde{T}$ of dimension 1, which is precisely the orthogonal complement of $\widetilde{K}$. This implies that each nowhere-zero vector field on $\mathbb{D}_\delta$ lifts compatibly to a vector field on $V \setminus \{0\}$ and also to a section of $\widetilde{T}|_V$. Finally notice that the gradient vector field $\overline{\mathrm{grad}_V f}$ can be obtained by lifting such a vector field, which we know from Lemma 8.3.1.

8.6 The Euler Obstruction via Morse Theory

This section is taken from [150], by J. Seade, M. Tibăr and A. Verjovsky. Here we show how stratified Morse theory yields to a clear understanding of what the invariant $\mathrm{Eu}_{f,V}(x)$ is for arbitrary functions with an isolated singularity. These results can also be deduced from Schürmann's book [138], and also from the work of D. Massey, [118, 119]. For this we recall the definition of complex stratified Morse singularities (see Goresky–MacPherson [73], p. 52).

Definition 8.6.1. Let V_α be a Whitney stratification of V and let $f : V \to \mathbb{C}$ be the restriction to V of a holomorphic function $\widehat{f} : \mathbb{C}^m \to \mathbb{C}$; assume for simplicity $0 = f(x)$. One says that $f : (V,x) \to (\mathbb{C},0)$ has a *stratified Morse critical point* at $x \in V$ if the dimension of the stratum V_α that contains x is ≥ 1, the restriction of f to V_α has a Morse singularity at x and f is general with respect to all other strata containing x in its closure, *i.e.*, $\mathrm{Ker}\, d\widehat{f}(x)$ is transverse in $\mathbb{C}^m$ to every limit of tangent spaces $T_{x_i}(V_\beta)$, for every stratum V_β such that $V_\alpha \subset \overline{V}_\beta$ and every sequence $x_i \in V_\beta$ converging to x.

We recall that every map-germ f on $(V,0)$ can be morsified, *i.e.*, approximated by Morse singularities. This is proved in [104] for f with an isolated singularity.

The theorem below is contained in [150]. Notice that 8.5.1 is included here.

Theorem 8.6.1. *Let f be a holomorphic function germ on $(V,0)$ with an isolated singularity (stratified critical point) at 0, restriction of a function $\widehat{f}$ on an open set in $\mathbb{C}^m$. Let $V_\alpha \subset V$ be the stratum that contains 0. Then:*

(1) *If $\dim V_\alpha < \dim V$ and $\operatorname{Ker} d\widehat{f}$ does not vanish on any generalized tangent space of the regular stratum (in particular if f is Morse at 0), then $\mathrm{Eu}_{f,V}(0) = 0$.*

(2) *If f has a stratified Morse singularity at $0 \in V_\alpha$ and $\dim V_\alpha = \dim V = n$, then $\mathrm{Eu}_{f,V}(0) = (-1)^n$.*

(3) *In general, the number of critical points of a Morsification of f in the regular part of V is $(-1)^{n+1}\mathrm{Eu}_{f,V}(0)$.*

Proof. Take a small enough ball $\mathbb{B}_\varepsilon$ in $\mathbb{C}^m$, centered at 0 and of radius $\varepsilon > 0$. Let v be the gradient vector field $\overline{\mathrm{grad}}_V f_V$ restricted to the sphere $V \cap \partial\mathbb{B}_\varepsilon$ and consider the lift $\widetilde{v}$ to the Nash blow-up $\widetilde{V}$ given by 8.1.1. By hypothesis the kernel of $d\widehat{f}$ does not vanish on any limit of tangent spaces at points in the regular stratum V_{reg}. Since $\widetilde{V}$ is obtained by attaching to TV_{reg} all limits of tangent spaces of points in V_{reg}, one has that if $0 \notin V_{\mathrm{reg}}$, then by the definition of stratified Morse points, the section $\widetilde{v}$ of $\widetilde{T}$ can be extended over $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$ without zeros, just as in the proof of 8.5.1. This proves the first statement in 5.4.

To prove the second statement of 5.4 we notice that in this case the variety V is locally isomorphic to $\mathbb{C}^n$ at 0, so its Nash transform is $\mathbb{C}^n$ and the Nash bundle $\widetilde{T}$ is the tangent bundle of $\mathbb{C}^n$. Hence, by definition, $\mathrm{Eu}_{f,V}(0)$ is the Poincaré–Hopf index at 0 of the gradient vector field $\overline{\mathrm{grad}} f = (\overline{\frac{\partial f}{\partial x_1}}, ..., \overline{\frac{\partial f}{\partial x_m}})$. This equals $(-1)^n$-times the Poincaré–Hopf index at 0 of the vector field $\mathrm{grad} f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_m})$. Hence our claim is equivalent to saying that the Milnor number μ of f is the degree of $\mathrm{grad} f$, which is Milnor's Theorem 7.2 in [121].

Finally statement (iii) is an immediate consequence of the previous two statements and the morsification theorem: perturb f to obtain a Morse function f_M on V. It is clear that $\mathrm{Eu}_{f,V}(0)$ equals the sum $\sum_j \mathrm{Eu}_{f_M,V}(q_j)$, where the sum runs over the Morse critical points q_j of f_M, since f and f_M can be assumed to coincide away from a small neighborhood of 0.

Remark: In [151] there are several formulae relating the invariant $\mathrm{Eu}_{f,V}(0)$ with other invariants of functions on singular varieties. In [77] this invariant is related with the Bruce–Roberts Milnor number defined in [38].

Chapter 9
Indices for 1-Forms

Abstract When considering smooth (real) manifolds, the tangent and cotangent bundles are isomorphic and it does not make much difference to consider either vector fields or 1-forms in order to define their indices and their relations with characteristic classes. When the ambient space is a complex manifold, this is no longer the case, but there are still ways for comparing indices of vector fields and 1-forms, and to use these to study Chern classes of manifolds. To some extent this is also true for singular varieties, but there are however important differences and each of the two settings has its own advantages.

In this chapter we briefly review the various indices of 1-forms on singular varieties through the light of the indices of vector fields discussed earlier. We define in that way the Schwartz index, the radial index, the GSV index, the homological index and the local Euler obstruction, and we study some of their relations and properties.

In this short presentation we include work done by various authors, particularly by W. Ebeling and S. Gusein-Zade, as well as ourselves in [36]. In the last section we discuss briefly the "indices of collections of 1-forms" introduced by W. Ebeling and S. Gusein-Zade: just as the index of a 1-form corresponds to the "top Chern class" (of a manifold or of a singular variety, in a sense that will be made precise in later chapters), so too the indices of collections of 1-forms correspond to other Chern numbers.

Let us mention that in his book [138], J. Schürmann introduces methods to studying singular varieties via micro-local analysis, and part of what we say below can also be considered in that framework.

9.1 Some Basic Facts About 1-Forms

In this section we study some basic facts about the geometry of 1-forms and the interplay between real and complex valued 1-forms on (almost) complex manifolds, which plays an important role in the sequel. The material here is all contained in the literature; we include it for completeness and to set up

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Lecture Notes in Mathematics 1987, DOI 10.1007/978-3-642-05205-7_9,

our notation with no possible ambiguities. We give precise references when appropriate.

Let M be an almost complex manifold of real dimension $2m > 0$. Let TM be its complex tangent bundle. We denote by T^*M the cotangent bundle of M, dual of TM; each fiber $(T^*M)_x$ consists of the $\mathbb{C}$-linear maps $TM_x \to \mathbb{C}$. We denote by $T_{\mathbb{R}}M$ the underlying real tangent bundle of M; it is a real vector bundle of fiber dimension $2m$, endowed with a canonical orientation. Its dual $T^*_{\mathbb{R}}M$ has fiber the $\mathbb{R}$-linear maps $(T_{\mathbb{R}}M)_x \to \mathbb{R}$.

Definition 9.1.1. Let A be a subset of M. By *a real (valued) 1-form* η on A we mean the restriction to A of a continuous section of the bundle $T^*_{\mathbb{R}}M$, *i.e.*, for each $x \in A$, η_x is an $\mathbb{R}$- linear map $(T_{\mathbb{R}}M)_x \to \mathbb{R}$. We usually drop the word "valued" here and speak only of real 1-forms on A. Similarly, *a complex 1-form* ω on A means the restriction to A of a continuous section of the bundle T^*M, *i.e.*, for each $x \in A$, ω_x is a $\mathbb{C}$-linear map $(TM)_x \to \mathbb{C}$.

Notice that the kernel of a real form η at a point x is either the whole fiber $(T_{\mathbb{R}}M)_x$ or a real hyperplane in it. In the first case we say that x is a singular point (or zero) of η. In the second case the kernel $ker\,\eta_x$ splits $(T_{\mathbb{R}}M)_x$ in two half spaces $(T_{\mathbb{R}}M^{\pm})_x$; in one of these the form takes positive values, in the other it takes negative values.

We recall that a vector field v in $\mathbb{R}^{2m}$ is radial at a point x if it is transverse to every sufficiently small sphere around x in $\mathbb{R}^{2m}$. The duality between real 1-forms and vector fields assigns to each tangent vector $\partial/\partial x_i$ the form dx_i (extending it by linearity to all tangent vectors). This motivates the following definition ([49, 50]):

Definition 9.1.2. A real 1-form η on M is *radial* (outwards-pointing) at a point $x \in M$ if, locally, it is dual over $\mathbb{R}$ to a radial outwards-pointing vector field at x. Inwards-pointing radial vector fields are defined similarly.

In other words, η is *radial* at a point x if it is everywhere positive when evaluated in some radial vector field at x. Thus, for instance, if for a fixed $x \in M$ we let $\rho_x(x)$ be the function $\|x - x\|^2$ (for some Riemannian metric), then its differential is a radial form.

Remark 9.1.1. The concept of radial forms was introduced in [49]. In [50] radial forms are defined using more relaxed conditions than we do here. However this is a concept taken from the corresponding notion of radial vector fields, so we use definition 9.1.2.

A complex 1-form ω on $A \subset M$ can be written in terms of its real and imaginary parts:

$$\omega \;=\; \mathrm{Re}\,(\omega) \;+\; i\,\mathrm{Im}\,(\omega).$$

Both $\mathrm{Re}\,(\omega)$ and $\mathrm{Im}\,(\omega)$ are real 1-forms, and the linearity of ω implies that for each tangent vector one has:

$$\mathrm{Im}\,(\omega)(v) \;=\; -\mathrm{Re}\,(\omega)(iv),$$

thus

$$\omega(v) \,=\, \mathrm{Re}\,(\omega)(v) \,-\, i\,\mathrm{Re}\,(\omega)(iv).$$

In other words the form ω is determined by its real part and one has a 1-to-1 correspondence between real and complex forms, assigning to each complex form its real part, and conversely, to a real 1-form η corresponds the complex form ω defined by:

$$\omega(v) \,=\, \eta(v) - i\eta(iv).$$

This statement (observed in [50],[73]) refines the obvious fact that a complex hyperplane P in $\mathbb{C}^m$, say defined by a linear form H, is the intersection of the real hyperplanes $\widehat{H} := \{\mathrm{Re}\, H = 0\}$ and $i\,\widehat{H}$. This justifies the following definition:

Definition 9.1.3. A complex 1-form ω is *radial* at a point $x \in M$ if its real part is radial at x.

Recall that the Euler class of an oriented vector bundle is the primary obstruction to constructing a nonzero section [153]. In the case of the bundle $T^*_{\mathbb{R}}M$, this class equals the Euler class of the underlying real tangent bundle $T_{\mathbb{R}}M$, since they are isomorphic. Thus, if M is compact then its Euler class evaluated on the orientation cycle of M gives the Euler–Poincaré characteristic $\chi(M)$. We can say this in different words: let η be a real 1-form on M with isolated (hence finitely many) singularities $x_1, \cdots, x_r$. At each x_i this 1-form defines a map, $\mathbb{S}_\varepsilon \xrightarrow{\eta/\|\eta\|} \mathbb{S}^{2m-1}$, from a small $(2m-1)$-sphere $\mathbb{S}_\varepsilon$ in M around x_i into the unit sphere in the fiber $(T^*_{\mathbb{R}}M)_x$. If we equip M and $T^*_{\mathbb{R}}M$ with the orientations induced by the almost complex structure on M, the degree of this map is the *Poincaré–Hopf* local index of η at x_i, that we may denote by $\mathrm{Ind}_{\mathrm{PH}}(\eta, x_i)$. Then the total index of η in M is by definition the sum of its local indices at the x_i and it equals $\chi(M)$. Its Poincaré dual class in $H^{2m}(M)$ is the Euler class of $T^*_{\mathbb{R}}M \cong T_{\mathbb{R}}M$.

More generally, if M is a compact C^∞ manifold of real dimension $2m$ with nonempty boundary ∂M and a complex structure in its tangent bundle, one can speak of real and complex valued 1-forms as above. Elementary obstruction theory (see [153]) implies that one can always find real and complex 1-forms on M with isolated singularities, all contained in the interior of M. In fact, if a real 1-form η is defined in a neighborhood of ∂M in M and it is nonsingular there, then we can always extend it to the interior of M with finitely many singularities, and its total index in M does not depend on the choice of the extension.

Definition 9.1.4. Let M be an almost complex manifold with boundary ∂M and let ω be a (real or complex) 1-form on M, nonsingular on a neighborhood of ∂M. The form ω is *radial* at the boundary if for each vector $v(x) \in TM$, $x \in \partial M$, which is normal to the boundary (for some metric), pointing outwards of M, one has $\mathrm{Re}\,\omega(v(x)) > 0$ (for real values 1-forms, $\mathrm{Re}\,\omega = \omega$).

By the theorem of Poincaré–Hopf for manifolds with boundary, if a real 1-form η is radial at the boundary and M is compact, then the total index of η is $\chi(M)$.

We now make similar considerations for complex 1-forms. We let M be a compact, C^∞ manifold of real dimension $2m$ (with or without boundary ∂M), with a complex structure in its tangent bundle TM. Let T^*M be as before, the cotangent bundle of M, *i.e.*, the bundle of complex valued continuous 1-forms. The top Chern class $c^m(T^*M)$ is the primary obstruction to constructing a section of this bundle, *i.e.*, if M has empty boundary, then $c^m(T^*M)$ is the number of points, counted with their local indices, of the zeroes of a section ω of T^*M (*i.e.*, a complex 1-form) with isolated singularities (*i.e.*, points where it vanishes). It is well-known that one has:

$$c^m(T^*M) \,=\, (-1)^m\, c^m(TM).$$

This corresponds to the fact that at each isolated singularity x_i of ω one has two local indices: one of them is the index of its real part defined as above, $\mathrm{Ind}_{\mathrm{PH}}(\mathrm{Re}\,\omega, x_i)$; the other is the degree of the map $\mathbb{S}_\varepsilon \xrightarrow{\omega/\|\omega\|} \mathbb{S}^{2m-1}$, that we denote by $\mathrm{Ind}_{\mathrm{PH}}(\omega, x_i)$. These two indices are related by the equality:

$$\mathrm{Ind}_{\mathrm{PH}}(\omega, x_i) \,=\, (-1)^m\, \mathrm{Ind}_{\mathrm{PH}}(\mathrm{Re}\,\omega, x_i),$$

and the index on the right corresponds to the local Poincaré–Hopf index of the vector field defined by duality near x_i. For example, the form $\omega = \sum z_i dz_i$ in $\mathbb{C}^m$ has index 1 at 0, while its real part $\sum(x_i dx_i - y_i dy_i)$ has index $(-1)^m$.

If we take M as above, compact and with possibly nonempty boundary, and ω is a complex 1-form with isolated singularities in the interior of M and radial on the boundary, then (by the previous considerations) the total index of ω in M is $(-1)^m\,\chi(M)$. We summarize some of the previous discussion in the following theorem ([49, 50]):

Theorem 9.1.1. *Let M be a compact, C^∞ manifold of real dimension $2m$ (with or without boundary ∂M), with a complex structure in its tangent bundle TM. Let $T^*_{\mathbb{R}}M$ and T^*M be as before, the bundles of real and complex valued continuous 1-forms on M, respectively. Then:*

(1) *Every real 1-form η on M determines a complex 1-form ω by the formula*

$$\omega(v) \,=\, \eta(v) - i\eta(iv)$$

so the real part of ω is $\mathrm{Re}\,\omega = \eta$.

(2) *The local Poincaré–Hopf indices at an isolated singularity of a complex 1-form and its real part are related by:*

$$\mathrm{Ind}_{\mathrm{PH}}(\omega, x_i) \,=\, (-1)^m\, \mathrm{Ind}_{\mathrm{PH}}(\mathrm{Re}\,\omega, x_i).$$

(3) *If a real 1-form on M is radial at the boundary ∂M, then its total Poincaré–Hopf index in M is $\chi(M)$. In particular, a radial real 1-form has local index 1.*

(4) *If a complex 1-form on M is radial at the boundary ∂M, then its total Poincaré–Hopf index in M is $(-1)^m\chi(M)$.*

Remark 9.1.2. One may consider frames of complex 1-forms on M instead of a single 1-form. This means considering sets of k complex 1-forms, whose singularities are the points where these forms become linearly dependent over $\mathbb{C}$. By definition (see [153]) the primary obstruction to constructing such a frame is the Chern class $c^{m-k+1}(T^*M)$, so these classes have an expression similar to 1.6 but using indices of frames of 1-forms. One always has $c^i(T^*M) = (-1)^i c^i(TM)$. Thus the Chern classes, and all the Chern numbers of M, can be computed using indices of either vector fields or 1-forms.

9.2 Radial Extension and the Schwartz Index

In the sequel we will be interested in considering forms defined on singular varieties in a complex manifold, so we introduce some standard notation. Let V be a reduced, equidimensional complex analytic space of dimension n in a complex manifold M of dimension m, endowed with a Whitney stratification $\{V_\alpha\}$ adapted to V, *i.e.*, V is a union of strata.

The following definition is an immediate extension for 1-forms of the corresponding (standard) definition for functions on stratified spaces in terms of its differential (c.f. [50, 73, 102]).

Definition 9.2.1. Let ω be a (real or complex) 1-form on V, *i.e.*, a continuous section of either $T^*_{\mathbb{R}}M|_V$ or $T^*M|_V$. *A singularity of* ω with respect to the Whitney stratification $\{V_\alpha\}$ means a point x where the kernel of ω contains the tangent space of the corresponding stratum.

This means that the pull back of the form to V_α vanishes at x.

In Sect. 1 we introduced the notion of radial forms, which is dual to the "radiality" for vector fields. We now extend this notion relaxing the condition of radiality in the directions tangent to the strata. From now on, unless it is otherwise stated explicitly, by a singularity of a 1-form on V we mean a singularity in the stratified sense, *i.e.*, in the sense of Definition 9.2.1.

Definition 9.2.2. Let ω be a (real or complex) 1-form on V. The form is *normally radial* at a point $x \in V_\alpha \subset V$ if it is radial when restricted to vectors which are not tangent to the stratum V_α. In other words, for every vector $v(y)$ tangent to M at a point $y \notin V_\alpha$, y sufficiently close to x and $v(y)$ pointing outwards a tubular neighborhood of the stratum V_α, $\mathrm{Re}\,\omega(v)$ is not zero and has constant sign for all such vectors.

Obviously a radial 1-form is also normally radial, since it is radial in all directions.

For each point x in a stratum V_α, one has a neighborhood U_x of x in M which is diffeomorphic to the product $U_\alpha \times \mathbb{D}_\alpha$, where $U_\alpha = U_x \cap V_\alpha$ and $\mathbb{D}_\alpha$ is a small disk in M transverse to V_α. Let π be the projection $\pi : U_x \to U_\alpha$ and p the projection $p : U_x \to \mathbb{D}_\alpha$. One has an isomorphism:

$$T^*U_x \cong \pi^*T^*U_\alpha \oplus p^*T^*\mathbb{D}_\alpha.$$

For a (real or complex) 1-form ω, to be normally radial at x says that up to a local change of coordinates in M, ω is the direct sum of the pull back of a (real or complex) form on U_α, *i.e.*, a section of the (real or complex) cotangent bundle T^*U_α, and a section of the (real or complex) cotangent bundle $T^*\mathbb{D}_\alpha$ which is a radial form in the disc.

We can proceed, for 1-forms, to the classical construction of *radial extension* introduced by M.-H. Schwartz in [139, 141] for stratified vector fields and frames. Locally, the construction can be described as follows. Firstly we consider real 1-forms. Let η be a 1-form on U_α, denote by $\widehat{\eta}$ its pull back to a section of $\pi^*T^*_{\mathbb{R}}U_\alpha$. This corresponds to the *parallel extension* of stratified vector fields done by Schwartz. Now consider the function ρ given by the square of the distance to the origin in $\mathbb{D}_\alpha$. The form $p^*d\rho$ on U_x vanishes on U_α and away from U_α its kernel is transverse to the strata of V by Whitney conditions.

The sum $\eta' = \widehat{\eta} + p^*d\rho$ defines a normally radial 1-form on U_x which coincides with η on U_α; away from U_α its kernel is transverse to the strata of V. Thus, if η is nonsingular at x, then η' is nonsingular everywhere on U_x. If η has an isolated singularity at $x \in V_\alpha$, then η' also has an isolated singularity there. In particular, if the dimension of the stratum V_α is zero then η' is a radial form in the sense of Sect. 1.

Following the terminology of [139,141] we say that the form η' is obtained from η by *radial extension.*

Since the index in M of a normally radial form is its index in the stratum times the index of a radial form in the disk $\mathbb{D}_\alpha$, we obtain the following important property of forms constructed by radial extension.

Proposition 9.2.1. *Let η be a real 1-form on the stratum V_α with an isolated singularity at a point x with local Poincaré–Hopf index* $\mathrm{Ind}_{\mathrm{PH}}(\eta, x, V_\alpha)$*. Let η' the 1-form on a neighborhood of x in M obtained by radial extension. Then the index of η in the stratum equals the index of η' in M:*

$$\mathrm{Ind}_{\mathrm{PH}}(\eta, x; V_\alpha) = \mathrm{Ind}_{\mathrm{PH}}(\eta', x; M).$$

Definition 9.2.3. The *Schwartz index* of the continuous real 1-form η at an isolated singularity $x \in V_\alpha \subset V$, denoted $\mathrm{Ind}_{\mathrm{Sch}}(\eta, x; V)$, is the Poincaré–Hopf index of the 1-form η' obtained from η by radial extension; or equivalently, if the stratum of x has dimension more than 0, $\mathrm{Ind}_{\mathrm{Sch}}(\eta, x; V)$ is the Poincaré–Hopf index at x of η in the stratum V_α.

If x is an isolated singularity of V then every 1-form on V must be singular at x since its kernel contains the "tangent space" of the stratum. In this case the index of the form in the stratum is defined to be 1, and this is consistent with the previous definition since in this case the radial extension of η is actually radial at x, so it has index 1 in the ambient space.

The previous process is easily adapted to give radial extension for complex 1-forms. Let ω be such a form on V_α; let η be its real part. We extend η as above, by radial extension, to obtain a real 1-form η' which is normally radial at x. Then we use statement (1) in Theorem 9.1.1 above to obtain a complex 1-form ω' on U_x that extends ω and is also normally radial at x. If we prefer, we can make this process in a different but equivalent way: first make a parallel extension of ω to U_x as above, using the projection π; denote by $\widehat{\omega}$ this complex 1-form. Now use (1) of Theorem 9.1.1) to define a complex 1-form $\widehat{d\rho}$ on U_x whose real part is $d\rho$, and take the direct sum of $\widehat{\omega}$ and $\widehat{d\rho}$ at each point to obtain the extension ω'. We say that ω' is obtained from ω by *radial extension.*

We have the equivalent of Proposition 9.2.1 for complex forms, modified with the appropriate signs:

$$(-1)^s \,\mathrm{Ind}_{\mathrm{PH}}(\omega, x, V_\alpha) \;=\; (-1)^m \,\mathrm{Ind}_{\mathrm{PH}}(\omega', x, M),$$

where $2s$ is the real dimension of V_α and $2m$ that of M.

Definition 9.2.4. The *Schwartz index* of the continuous complex 1-form ω at an isolated singularity $x \in V_\alpha \subset V$, denoted $\mathrm{Ind}_{\mathrm{Sch}}(\omega, x, V)$, is $(-1)^n$-times the index of its real part:

$$\mathrm{Ind}_{\mathrm{Sch}}(\omega, x; V) \;=\; (-1)^n \mathrm{Ind}_{\mathrm{Sch}}(\mathrm{Re}\,\omega, x; V).$$

9.3 Local Euler Obstruction of a 1-Form and the Proportionality Theorem

In this section, we are concerned with a local situation, so we take the n-dimensional complex variety V to be embedded in an open ball $\mathbb{B} \subset \mathbb{C}^m$ centered at the origin 0.

The local Euler obstruction of a 1-form was introduced in [52] in analogy with the case of vector fields discussed in the previous chapter. Let us recall its definition.

On the regular part of V one has the map $\sigma : V_{\mathrm{reg}} \to G(n, m)$ into the Grassmannian of complex n-planes in $\mathbb{C}^m$, that assigns to each point the corresponding tangent space of V_{reg}. Let us recall (Sect. 8.1) that one has the Nash bundle $\widetilde{T} \xrightarrow{p} \widetilde{V}$, restriction to the Nash blow-up $\widetilde{V}$ of the tautological bundle over $\mathbb{B} \times G(n,m)$.

The corresponding dual bundles of complex and real 1-forms are denoted by $\widetilde{T}^* \xrightarrow{p} \widetilde{V}$ and $\widetilde{T}^*_{\mathbb{R}} \xrightarrow{p} \widetilde{V}$, respectively. Observe that a point in $\widetilde{T}^*$ is a triple (x, P, ω) where x is in V, P is an n-plane in the tangent space $T_x\mathbb{B}$ which is limit of a sequence $\{(TV_{\text{reg}})_{x_i}\}$, where the x_i are points in the regular part of V converging to x, and ω is a $\mathbb{C}$-linear map $P \to \mathbb{C}$. (Similarly for $\widetilde{T}^*_{\mathbb{R}}$.)

Let us denote by ρ the function given by the square of the distance to 0. We recall that MacPherson in [117] observed that the Whitney condition (a) implies that the pull-back of the differential $d\rho$ defines a never-zero section $\widetilde{d\rho}$ of $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(\mathbb{S}_\varepsilon \cap V) \subset \widetilde{V}$, where $\mathbb{S}_\varepsilon$ is the boundary of a small ball $\mathbb{B}_\varepsilon$ in $\mathbb{B}$ centered at 0. The obstruction to extending $\widetilde{d\rho}$ as a never-zero section of $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(\mathbb{B}_\varepsilon \cap V) \subset \widetilde{V}$ is a cohomology class in $H^{2n}(\nu^{-1}(\mathbb{B}_\varepsilon \cap V), \nu^{-1}(\mathbb{S}_\varepsilon \cap V); \mathbb{Z})$, and MacPherson defined *the local Euler obstruction* $\mathrm{Eu}_V(0)$ of V at 0 to be the integer obtained by evaluating this class on the orientation cycle $[\nu^{-1}(\mathbb{B}_\varepsilon \cap V), \nu^{-1}(\mathbb{S}_\varepsilon \cap V)]$.

More generally, given a section η of $T^*_{\mathbb{R}}\mathbb{B}|_A$, $A \subset V$, there is a canonical way of constructing a section $\tilde{\eta}$ of $\widetilde{T}^*_{\mathbb{R}}|_{\tilde{A}}$, $\tilde{A} = \nu^{-1}A$, which is described in the following. The same construction works for complex forms. First, taking the pull-back $\nu^*\eta$, we get a section of $\nu^*T^*_{\mathbb{R}}\mathbb{B}|_V$. Then $\tilde{\eta}$ is obtained by projecting $\nu^*\eta$ to a section of $\tilde{T}^*_{\mathbb{R}}$ by the canonical bundle homomorphism

$$\nu^*T^*_{\mathbb{R}}\mathbb{B}|_V \longrightarrow \tilde{T}^*_{\mathbb{R}}.$$

Thus the value of $\tilde{\eta}$ at a point (x, P) is simply the restriction of the linear map $\eta(x) : (T_{\mathbb{R}}\mathbb{B})_x \to \mathbb{R}$ to P. We call $\tilde{\eta}$ the *canonical lifting* of η.

By the Whitney condition (a), if $a \in V_\alpha$ is the limit point of the sequence $\{x_i\} \in V_{\text{reg}}$ such that $P = \lim(TV_{\text{reg}})_{x_i}$ and if the kernel of η is transverse to V_α, then the linear form $\widetilde{\eta}$ will be nonvanishing on P. Thus, if η has an isolated singularity at the point $0 \in V$ (in the stratified sense), then we have a never-zero section $\widetilde{\eta}$ of the dual Nash bundle $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(\mathbb{S}_\varepsilon \cap V) \subset \widetilde{V}$. Let $o(\eta) \in H^{2n}(\nu^{-1}(\mathbb{B}_\varepsilon \cap V), \nu^{-1}(\mathbb{S}_\varepsilon \cap V); \mathbb{Z})$ be the cohomology class of the obstruction cycle to extend this to a section of $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(\mathbb{B}_\varepsilon \cap V)$. Then define (c.f. [32, 50]):

Definition 9.3.1. The *local Euler obstruction* of the real differential form η at an isolated singularity is the integer $\mathrm{Eu}_V(\eta, 0)$ obtained by evaluating the obstruction cohomology class $o(\eta)$ on the orientation cycle $[\nu^{-1}(\mathbb{B}_\varepsilon \cap V), \nu^{-1}(\mathbb{S}_\varepsilon \cap V)]$.

MacPherson's local Euler obstruction $\mathrm{Eu}_V(0)$ corresponds to taking the differential of the square of the function distance to 0.

In the complex case, one can perform the same construction, using the corresponding complex bundles. If ω is a complex differential form, section of

$T^*\mathbb{B}|_A$ with an isolated singularity, one can define the local Euler obstruction $\mathrm{Eu}_V(\omega, 0)$. Notice that it is equal to that of its real part up to sign:

$$\mathrm{Eu}_V(\omega, 0) \;=\; (-1)^n \mathrm{Eu}_V(\mathrm{Re}\,\omega, 0). \tag{9.3.1}$$

This is an immediate consequence of the relation between the Chern classes of a complex vector bundle and those of its dual.

We note that the idea of considering the (complex) dual Nash bundle was already present in [134], where Sabbah introduces a local Euler obstruction $\check{\mathrm{Eu}}_V(0)$ that satisfies $\check{\mathrm{Eu}}_V(0) = (-1)^n \mathrm{Eu}_V(0)$. See also [137], Sect. 5.2.

We also notice (see [151, Corollary 3.2] and [52, Proposition 4]) that if $(V, 0)$ is a reduced complex analytic germ with an isolated singularity at 0 and f is a holomorphic function on V with an isolated singularity at 0, then one has

$$\mathrm{Eu}_V(df, 0) = (-1)^{\dim V}[\chi(F_\ell) - \chi(F_f)],$$

where F_* denotes the Milnor fiber and ℓ is a generic linear function on V (so that F_ℓ is the *complex link* of 0 in V in the sense of [73]).

Just as for vector fields (see Chap. 8) one has in this situation the following:

Theorem 9.3.2. *Let $V_\alpha \subset V$ be the stratum containing* 0*,* $\mathrm{Eu}_V(0)$ *the local Euler obstruction of V at* 0 *and ω a (real or complex) 1-form on V_α with an isolated singularity at* 0*. Then the local Euler obstruction of the radial extension ω' of ω and the Schwartz index of ω at* 0 *are related by the following proportionality formula:*

$$\mathrm{Eu}_V(\omega', 0) \;=\; \mathrm{Eu}_V(0) \cdot \mathrm{Ind}_{\mathrm{Sch}}(\omega, 0; V).$$

The Theorem can be proved by one of the two ways we used for proving Theorem 3.6.1 or Theorem 8.1.2 (for details see [36]).

9.4 The Radial Index

In [50, 51] the authors introduced an index of 1-forms with isolated singularities on (real) analytic varieties, that they called *radial index* in analogy with the previously defined index for vector fields (see [6, 49, 96, 148] or Chaps. 2 and 4 above). This index measures the "lack of radiality" of such a 1-form. The corresponding notion for complex 1-forms with isolated singularities on complex analytic varieties was introduced in [57] (see also [52]). Here we briefly explain this index, essentially following [57].

Let $(V, 0) \subset (\mathbb{C}^m, 0)$ be a germ of a purely n-dimensional complex analytic variety with an isolated singularity at the origin 0. Let ω be a continuous 1-form on V with an isolated singularity at the origin 0, so ω is a continuous, nowhere-vanishing section of the complex cotangent bundle of $V \setminus \{0\}$.

Let us fix a radial vector field $v_{\rm rad}$ on $(V,0)$, *e.g.*, the gradient on the smooth part of V of the real valued function $\|z\|$ with respect to a Riemannian metric.

Definition 9.4.1. A real (or complex) 1-form on V is radial at 0 if, near the origin, its value on the radial vector field $v_{\rm rad}$ has positive real part at each point in a punctured neighborhood of the origin 0 in V. We denote such a form by $\omega_{\rm rad}$.

The space of such 1-forms is connected.

Let ω_1 and ω_2 be 1-forms on $(V,0)$ with isolated singularities at the origin. Choose $\varepsilon > \varepsilon' > 0$ sufficiently small, let $\mathbf{K}_\varepsilon = V \cap \mathbb{S}_\varepsilon$ and $\mathbf{K}_{\varepsilon'} = V \cap \mathbb{S}_{\varepsilon'}$ be the corresponding links, and let Z be the cylinder $V \cap [B_\varepsilon \setminus \mathrm{Int}(B_{\varepsilon'})]$, where B_ρ is the ball of radius ρ around the origin 0 in $\mathbb{C}^m$, $\mathbb{S}_\rho$ is its boundary. Let $\widetilde{\omega}$ be a 1-form on the cylinder Z which coincides with ω_1 in a neighborhood of $\mathbf{K}_\varepsilon$ and with ω_2 in a neighborhood of $\mathbf{K}_{\varepsilon'}$ and which has isolated singular points $q_1, \dots, q_s$ inside Z. The sum $d(\omega_1, \omega_2)$ of the (usual) local indices $\mathrm{Ind}(\widetilde{\omega}, q_i)$ of the form $\widetilde{\omega}$ at these points depends only on the forms ω_1 and ω_2 and will be called *the difference* of these forms. One has $d(\omega_1, \omega_2) = -d(\omega_2, \omega_1)$.

Definition 9.4.2. The *radial index* at 0, $\mathrm{Ind}_{\rm rad}(\omega, 0; V)$, of the 1-form ω on V is defined by

$$\mathrm{Ind}_{\rm rad}\,(\omega, 0; V) \;=\; (-1)^n + d(\omega, \omega_{\rm rad}).$$

Remark 9.4.1. Notice this definition is similar to that of the Schwartz index of vector fields given in the first section of Chap. 2. Also notice that the index of a radial 1-form $\omega_{\rm rad}$ is equal to $(-1)^n$. The sign is chosen so that this index coincides with the usual one if V is smooth at 0.

Remark 9.4.2. We know from Sect. 1 in this chapter that there is a one-to-one correspondence between complex 1-forms on a complex analytic manifold $V \setminus \{0\}$ and real 1-forms on it. The radial index of a complex 1-form can be expressed through the corresponding index of its real part, defined in [50,51], and viceversa. As before, the radial index $\mathrm{Ind}_{\rm rad}\,(\omega, 0; V)$ of a complex 1-form ω equals $(-1)^n$-times the radial index of its real part.

Example 9.4.1. Let ω be a holomorphic 1-form on a curve singularity $(C,0)$ with $C = \cup_{i=1}^r C_i$, where C_i are the irreducible components of C. Let t_i be a uniformization parameter on the component C_i and let the restriction $\omega_{|C_i}$ be of the form

$$(a_i t_i^{m_i} + \text{ terms of higher degree})\, dt_i\ ,\ a_i \neq 0.$$

Then $\mathrm{Ind}_{\rm rad}\,(\omega_{|C_i}, 0; C_i) = m_i$. Therefore $d(\omega_{|C_i}, \omega_{\mathrm{rad}|C_i}) = m_i + 1$, $d(\omega, \omega_{\rm rad}) = \sum_{i=1}^r (m_i + 1)$, $\mathrm{Ind}_{\rm rad}\,(\omega, 0; V) = \sum_{i=1}^r m_i + (r-1)$.

Remark 9.4.3. The radial index obviously satisfies a law of conservation of number: if ω' is a 1-form on V close to ω, then,

$$\mathrm{Ind}_{\mathrm{rad}}(\omega,0;V) = \mathrm{Ind}_{\mathrm{rad}}(\omega',0;V) + \sum \mathrm{Ind}_{\mathrm{rad}}(\omega',x;V),$$

where the sum on the right hand side is over all those points x in a small punctured neighborhood of the origin 0 in V where the form ω' vanishes (this follows from the fact that $d(\omega_1,\omega_3) = d(\omega_1,\omega_2) + d(\omega_2,\omega_3)$). This stability property of the index will be used in the last section.

Remark 9.4.4. In [52] Ebeling and Gusein-Zade define the radial index in a more general setting, analogous to the way we defined this index for vector fields in 2.4.2, and the theorem below holds in that more general setting.

Recall that if M is a compact complex manifold and ω is a complex 1-form on it with isolated singularities, then one has the usual local Poincaré–Hopf index at each singular point. These add up to the total index of the 1-form, which equals the Euler–Poincaré characteristic of M:

$$\mathrm{Ind}_{\mathrm{PH}}(\omega;M) = (-1)^m \chi(M),$$

independently of the 1-form. This is actually a special case of the last statement in Theorem 9.1.1, taking the boundary to be empty. We also know that in the case of vector fields, the work of M.-H. Schwartz shows that if V is now a compact complex analytic singular variety and v is a stratified vector field on V obtained by radial extension, then its total index equals the Euler–Poincaré characteristic $\chi(V)$. As explained in Chap. 2, this result extends easily to arbitrary vector fields on V, provided they are stratified and with isolated singularities, using the radial index.

These results extend naturally to 1-forms on singular varieties, as observed by W. Ebeling and S. Gusein-Zade in [50, 51]. One gets:

Theorem 9.4.1. *Let V be a compact complex analytic variety of dimension n and ω a differential complex 1-form on V with isolated singularities. Let $\mathrm{Ind}_{\mathrm{rad}}(\omega;V)$ denote the total radial index of ω, i.e., the sum of all its local radial indices at its singular points. Then:*

$$\mathrm{Ind}_{\mathrm{rad}}(\omega;V) = (-1)^n \chi(V).$$

9.5 The GSV Index

We look first at the case studied by Ebeling and Gusein-Zade, *i.e.*, when the variety V is an isolated complete intersection germ; then we envisage the case when V has nonisolated singularities, following [36].

9.5.1 Isolated Singularity Case

We recall (Chap. 3) that the GSV index of a continuous vector field v on an isolated complete intersection germ $(V,0)$ is the degree of the map:

$$\phi_v = (v, \overline{\text{grad}} f_1, ..., \overline{\text{grad}} f_k) : \mathbf{K} \to W_{k+1}(n+k),$$

from the link $\mathbf{K}$ of V into the Stiefel manifold $W_{k+1}(n+k)$, where $(f_1, \ldots, f_k)$ are functions that define the ICIS germ $(V,0)$ and v is assumed to be nonzero away from 0. As we know, this index equals the Poincaré–Hopf index of an extension of v to a Milnor fiber $\mathbf{F}$. The analogous index was defined in [50,51] for 1-forms on V.

Let ω be a complex-valued 1-form on V with an isolated singularity at 0. Then its GSV index equals the degree of the map:

$$\psi_\omega = (\omega, df_1, \ldots, df_k) : \mathbf{K} \to W^*_{k+1}(n+k),$$

where $W^*_{k+1}(n+k)$ denotes the bundle associated to the cotangent bundle $T^*(\mathbb{C}^{n+k})|_V$ with fiber the corresponding Stiefel manifold of complex orthonormal $(k+1)$-frames in the dual of $\mathbb{C}^{n+k}$. As noticed in [50, 51], this index equals the Poincaré–Hopf index of the 1-form on a Milnor fiber of f, *i.e.*, it equals the number of zeroes, counted with multiplicities, of any extension of ω to a Milnor fiber $V_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon$ of $(V,0)$. (The proof is similar to that for vector fields given in Chap. 3.)

A remarkable difference of this index with the analogous one for vector fields was observed in [50, 51]: if the differential 1-form is holomorphic, then its index can be regarded as an intersection number of complex manifolds, while for vector fields, the definition of the GSV index involves the conjugate gradient vector fields, which are anti-holomorphic. Thence, in the case of holomorphic 1-forms we can use powerful techniques of algebraic geometry to compute its index. More precisely, assume the 1-form ω is holomorphic, and let I be the ideal in $\mathcal{O}_{\mathbb{C}^{n+k},0}$ generated by $f_1, \ldots, f_k$ and the $(k+1) \times (k+1)$-minors of the matrix:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+k}} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_{n+k}} \\ A_1 & \cdots & A_{n+k} \end{pmatrix}.$$

Then one has the following theorem of W. Ebeling and S. Gusein-Zade (see [50, 51]):

Theorem 9.5.1.

$$\text{Ind}_{\text{GSV}}(\omega, 0; V) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+k},0}/I.$$

This formula extends the one obtained by Lê D.T. and G.-M. Greuel for the case when ω is the differential of a function ([74, 101]); in that case the formula gives the Milnor number of the function and is known as the *Lê-Greuel formula for the Milnor number.*

Remark 9.5.1. The above index can be regarded in the more general setting of residues of Chern classes defined by a finite number of holomorphic sections. See [160], where various expressions of the residues as in Sect. 1.6.6 are given. The formula (9.5.1) is a particular case of the algebraic expression there.

It is clear that the GSV index satisfies the same law of conservation of number satisfied by the radial index: if ω' is a 1-form on V close to ω, then:

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, 0; V) = \mathrm{Ind}_{\mathrm{GSV}}(\omega', 0; V) + \sum \mathrm{Ind}_{\mathrm{rad}}(\omega', x; V),$$

where the sum on the right hand side is over all those points x in a small punctured neighborhood of the origin 0 in V where the form ω' vanishes. This implies:

Proposition 9.5.1. *Let $\mu(V, 0)$ be the Milnor number of the isolated complete intersection singularity $(V, 0)$. For any 1-form ω on $(V, 0)$ with an isolated singularity at the origin 0 one has*

$$\mu(V, 0) \;=\; \mathrm{Ind}_{\mathrm{GSV}}\,(\omega, 0; V) \;-\; \mathrm{Ind}_{\mathrm{rad}}\,(\omega, 0; V)\,.$$

9.5.2 Nonisolated Singularity Case

If V has nonisolated singularities one may not have a Milnor fibration in general, but one does if V has a Whitney stratification satisfying Thom's a_f-condition, for the functions that define V (c.f. [34, 103, 107]).

Let $(V, 0)$ be a complete intersection of complex dimension n defined in an open ball $\mathbb{B}$ in $\mathbb{C}^{n+k}$ by functions $f = (f_1, \cdots, f_k)$, and assume 0 is a singular point of V (not necessarily an isolated singularity). As before, we endow $\mathbb{B}$ with a Whitney stratification $\{V_\alpha\}$ adapted to V, and we assume the stratification has the Thom property relatively to f. In particular, if $k = 1$ then we always have such stratifications, by [82]. For $k > 1$ we must assume such a stratification exists.

Let ω be as before, a (real or complex) 1-form on $\mathbb{B}$, and assume its restriction to V has an isolated singularity at 0. The kernel of $\omega(0)$ contains the tangent space of the stratum V_α containing 0, but if $x \neq 0$, the kernel of $\omega(x)$ is transverse to the stratum containing x. Now let $\mathbf{F} = \mathbf{F}_t$ be a Milnor fiber of V, *i.e.*, $\mathbf{F} = f^{-1}(t) \cap \mathbb{B}_\varepsilon$, where $\mathbb{B}_\varepsilon$ is a sufficiently small ball in $\mathbb{B}$ around 0 and $t \in \mathbb{C}^k$ is a regular value of f with $\|t\|$ sufficiently small with respect to ε. Notice that the a_f-condition implies that for every sequence

t_n of regular values converging to 0, and for every sequence $\{x_n\}$ of points in the corresponding Milnor fibers converging to a point $x \in V$ so that the sequence of tangent spaces $\{(T\mathbf{F}_{t_n})_{x_n}\}$ has a limit T, one has that T contains the space $(TV_\alpha)_x$, tangent to the stratum that contains x. By transversality this implies that choosing the regular value t sufficiently close to 0 we can assure that the kernel of ω is transverse to the Milnor fiber at every point in its boundary $\partial\mathbf{F}$. Thus its pull-back to $\mathbf{F}$ is a 1-form on this smooth manifold, and it is never-zero on its boundary, thence ω has a well-defined Poincaré–Hopf index in $\mathbf{F}$ as in Sect. 1. This index depends only on the restriction of ω to V and on the topology of the Milnor fiber $\mathbf{F}$, which is well-defined once we fix the defining function f (which is assumed to satisfy the a_f-condition for some Whitney stratification).

Definition 9.5.1. *The GSV index* of ω at $0 \in V$ relative to f, $\mathrm{Ind}_{\mathrm{GSV}}(\omega, 0; f)$, is the Poincaré–Hopf index of ω in $\mathbf{F}$.

In other words this index measures the number of points (counted with signs) in which a generic perturbation of ω is tangent to $\mathbf{F}$. In fact the inclusion $\mathbf{F} \stackrel{i}{\hookrightarrow} M$ pulls the form ω to a section of the (real or complex, as the case may be) cotangent bundle of $\mathbf{F}$, which is never-zero near the boundary $\partial\mathbf{F}$ since ω has an isolated singularity at 0 and, by hypothesis, the map f satisfies the Thom a_f-condition. One gets the following result, which is due to W. Ebeling and S. Gusein-Zade [50] when V has an isolated singularity:

Theorem 9.5.2. *If the form ω is real then*

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, 0; f) \,=\, e(\mathbf{F}; \omega)[\mathbf{F}], \tag{9.5.3}$$

where $e(\mathbf{F}; \omega) \in H^{2n}(\mathbf{F}, \partial\mathbf{F})$ is the Euler class of the real cotangent bundle $T^*_{\mathbb{R}}\mathbf{F}$ relative to the section defined by ω on the boundary, and $[\mathbf{F}]$ is the orientation cycle of the pair $(\mathbf{F}, \partial\mathbf{F})$. If ω is a complex form, then one has:

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, 0; f) \,=\, c^n(T^*\mathbf{F}; \omega)[\mathbf{F}], \tag{9.5.4}$$

where $c^n(T^*\mathbf{F}; \omega)$ is the top Chern class of the cotangent bundle of $\mathbf{F}$ relative to the form ω on the boundary $\partial\mathbf{F}$.

Notice this is analogous to the construction done in Sect. 1.3.2. In this case one can, alternatively, express the index as the relative Chern class:

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, 0; f) \,=\, c^n(T^*M|_{\mathbf{F}}; \Omega)\,[\mathbf{F}], \tag{9.5.5}$$

where Ω is the frame of $k+1$ complex 1-forms on the boundary of $\mathbf{F}$ defined by

$$\Omega \,=\, (\omega, df_1, df_2, \cdots, df_k),$$

since the forms $(df_1, \cdots, df_k)$ are linearly independent everywhere on $\mathbf{F}$. Notice that if the form ω is holomorphic, then this index is necessarily non-negative because it can be regarded as an intersection number of complex submanifolds. For every complex 1-form one has:

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega, 0; f) \,=\, (-1)^n \mathrm{Ind}_{\mathrm{GSV}}(\mathrm{Re}\,\omega, 0; f).$$

We remark that if V has an isolated singularity at 0, then this is the index defined in [50], *i.e.*, the degree of the map from the link $\mathbf{K}$ of V into the Stiefel manifold of complex (k+1)-frames in the dual $(\mathbb{C}^*)^{n+k}$ given by the map $(\omega, df_1, \cdots, df_k)$. Also notice that this index is somehow dual to the index envisaged in 3.5 for vector fields, which is related to the top Fulton–Johnson class of singular hypersurfaces, as we shall see in Chap. 11.

So, given the (nonisolated) complete intersection singularity $(V, 0)$ and a (real or complex) 1-form ω on V with an isolated singularity at 0, one has three different indices: the Euler obstruction (Sect. 9.3 in this chapter), the GSV index just defined and the index of its pull back to a 1-form on the stratum containing 0. One also has the index of the form in the ambient manifold M. For differential forms obtained by radial extension, the index in the stratum equals its index in M, and this is by definition the Schwartz index. The following proportionality theorem is analogous to the one in [34] for vector fields that we discussed in Chap. 3 above.

Theorem 9.5.6. *Let ω be a (real or complex) 1-form on the stratum V_α of 0 with an isolated singularity at 0. Then the GSV index of its radial extension ω' is proportional to the Schwartz index, the proportionality factor being the Euler–Poincaré characteristic of the Milnor fiber $\mathbf{F}$:*

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega', 0; f) \,=\, \chi(\mathbf{F}) \cdot \mathrm{Ind}_{\mathrm{Sch}}(\omega, 0; V).$$

The proof is similar to that of Theorem 9.3.2.

Remark 9.5.2. We notice that Theorems 9.3.2 and 9.5.6 can also be proved using the stability of the index under perturbations, just as we did for vector fields. More precisely, one can easily show that the Euler obstruction $\mathrm{Eu}_V(\omega, x)$ and the GSV index are stable when we perturb the 1-form (or the vector field) in the stratum and then extend it radially; then the sum of the indices at the singularities of the new 1-form (vector field) give the corresponding index for the original singularity. This implies the proportionality of the indices.

9.6 The Homological Index

This and the following sections are taken from [57]. Here we introduce the homological index of a 1-form on a complex analytic variety with an isolated singular point. This is analogous to, and inspired by, the homological index for vector fields defined in [68] and discussed in Chap. 7. As in the case of vector fields, when the ambient space is an ICIS, this index coincides with the previously defined GSV index of 9.5.1.

Let $(V, 0) \subset (\mathbb{C}^m, 0)$ be an arbitrary germ of an analytic variety of pure dimension n with an isolated singular point at the origin (not necessarily a complete intersection). Given a holomorphic form ω on $(V, 0)$ with an isolated singularity, we consider the complex $(\Omega^\bullet_{V,0}, \wedge\omega)$:

$$0 \longrightarrow \mathcal{O}_{V,0} \longrightarrow \Omega^1_{V,0} \longrightarrow \cdots \longrightarrow \Omega^n_{V,0} \longrightarrow 0,$$

where $\Omega^i_{V,0}$ are the modules of germs of Kähler differential forms on $(V, 0)$ as in 7.1.1, and the arrows are given by the exterior product by the form ω.

This complex is the dual of the Koszul complex considered in Chap. 7, and it was used by G.-M. Greuel in [74] for complete intersections. The sheaves $\Omega^i_{V,0}$ are coherent sheaves and the homology groups of the complex $(\Omega^\bullet_{V,0}, \wedge\omega)$ are concentrated at the origin and therefore are finite dimensional.

Definition 9.6.1. The *homological index* $\mathrm{Ind}_{\mathrm{hom}}(\omega, 0; V)$ of the 1-form ω on $(V, 0)$ is $(-1)^n$ times the Euler characteristic of the above complex:

$$\mathrm{Ind}_{\mathrm{hom}}(\omega, 0; V) = \sum_{i=0}^{n} (-1)^{n-i} h_i(\Omega^\bullet_{V,0}, \wedge\omega), \tag{9.6.1}$$

where $h_i(\Omega^\bullet_{V,0}, \wedge\omega)$ is the dimension of the corresponding homology group as a vector space over $\mathbb{C}$.

Theorem 9.6.2. *Let ω be a holomorphic 1-form on V with an isolated singularity at the origin* 0*.*

(1) *If V is smooth, then* $\mathrm{Ind}_{\mathrm{hom}}(\omega, 0; V)$ *equals the usual local index of the 1-form ω.*

(2) *The homological index satisfies the law of conservation of number: if ω' is a holomorphic 1-form on V close to ω (in the space of all holomorphic* 1*-forms on V), then:*

$$\mathrm{Ind}_{\mathrm{hom}}(\omega, 0; V) = \mathrm{Ind}_{\mathrm{hom}}(\omega', 0; V) + \sum \mathrm{Ind}_{\mathrm{hom}}(\omega', x, V),$$

where the sum on the right hand side is over all those points x in a small punctured neighborhood of the origin 0 *in V where the form ω' vanishes.*

(3) *If* $(V, 0)$ *is an isolated complete intersection singularity, then the homological index* $\mathrm{Ind}_{\mathrm{hom}}(\omega, 0; V)$ *coincides with the GSV index* $\mathrm{Ind}_{\mathrm{GSV}}(\omega, 0; V)$.

The proof of this theorem is an exercise using [64] and [74]. In fact, statement (1) is straightforward and it is a special case of statement (3). Statement (2) is a particular case of the main theorem in [64], which is Theorem 7.1.4 above. For statement (3) we notice that on an isolated complete intersection singularity $(V, 0))$ the index $\mathrm{Ind}_{\mathrm{GSV}}\omega$ also satisfies the law of conservation of number and coincides with the homological index $\mathrm{Ind}_{\mathrm{hom}}\,\omega$ on smooth varieties. This implies that the difference between these two indices is a locally constant, and therefore constant, function on the space of 1-forms on $(V, 0)$ with an isolated singular point at the origin. Therefore it suffices to prove (3) for $\omega = df$ where f is a holomorphic function on $(V, 0)$ with an isolated critical point at the origin. Then Lemma 1.6 in [74] implies that the homology groups of the complex $(\Omega^\bullet_{V,0}, \wedge df)$ vanish in dimensions $i = 0, 1, ..., n-1$. The statement then follows from the Remark following Lemma 5.3 of [74] (see also [51]). □

Remark 9.6.1. The minimal value of the homological index $\mathrm{Ind}_{\mathrm{hom}}(\omega, 0; V)$ is attained by restrictions to V of generic 1-forms on $\mathbb{C}^m$ which do not vanish at the origin. The subset of forms with this index in $\Omega^1_{V,0}$ is open, dense and connected. Moreover, each 1-form ω can be approximated by a 1-form, the index of which at the origin coincides with the minimal one and all its zeros on $V \setminus \{0\}$ are nondegenerate. This approximation can be chosen of the form $\omega + \varepsilon d\ell$ for a linear function ℓ.

Remark 9.6.2. We notice that one has an invariant for functions on $(V, 0)$ with an isolated singularity at the origin defined by $f \mapsto \mathrm{Ind}_{\mathrm{hom}}\,df$. By the theorem above, if $(V, 0)$ is an isolated complete intersection singularity, this invariant counts the number of critical points of the function f on a Milnor fiber.

Remark 9.6.3. Let $(C, 0)$ be an analytic curve singularity and let $(\bar{C}, \bar{0})$ be its normalization. Let $\tau = \dim \mathrm{Ker}(\Omega^1_{C,0} \to \Omega^1_{\bar{C},\bar{0}})$, $\lambda = \dim_{\mathbb{C}}(\omega_{C,0}/c(\Omega^1_{C,0}))$, where $\omega_{C,0}$ is the dualizing module of Grothendieck, $c : \Omega^1_{C,0} \to \omega_{C,0}$ is the class map (see [41]). In the article [123] of D. Mond and D. Van Straten there is considered a Milnor number of a function f on a curve singularity introduced by V. Goryunov. One can see that this Milnor number can be defined for a 1-form ω with an isolated singularity on $(C, 0)$ as well (as $\dim_{\mathbb{C}}(\omega_{C,0}/\omega \wedge \mathcal{O}_{C,0})$) and is equal to $\mathrm{Ind}_{\mathrm{hom}}\,\omega + \lambda - \tau$.

9.7 On the Milnor Number of an Isolated Singularity

We know from 9.5.1 that if $(V, 0)$ is an ICIS germ and ω is a 1-form on it with an isolated singularity at 0, then the Milnor number of $(V, 0)$ equals the difference of the GSV and radial indices,

$$\mu(V,0) \;=\; \mathrm{Ind}_{\mathrm{GSV}}(\omega,0;V) - \mathrm{Ind}_{\mathrm{rad}}(\omega,0;V).$$

If the germ $(V,0)$ is an isolated singularity but is not complete intersection, one does not have a Milnor number in general, neither one has a GSV index for forms nor vector fields. However the radial index is always defined, and so is the homological index if the forms are holomorphic; we also know that the homological index coincides with the GSV index if $(V,0)$ is a complete intersection germ. The laws of conservation of numbers for the homological and the radial indices of 1-forms, together with the fact that these two indices coincide on smooth varieties imply that their difference is a locally constant, and therefore constant, function on the space of 1-forms on V with isolated singularities at the origin. Therefore one has the following statement from [57]:

Proposition 9.7.1. *Let $(V,0)$ be a germ of a complex analytic space of pure dimension n with an isolated singular point at the origin. Then the difference*

$$\nu(V,0) = \mathrm{Ind}_{\mathrm{hom}}\,(\omega,0;V) - \mathrm{Ind}_{\mathrm{rad}}\,(\omega,0;V)$$

between the homological and the radial indices does not depend on the 1-form ω.

This proposition, together with 9.5.1, allows us to think of $\nu(V,0)$ as a generalized Milnor number of the singularity $(V,0)$.

There are other invariants of isolated singularities of complex analytic varieties which coincide with the Milnor number for isolated complete intersection singularities. One of them is $(-1)^n$ times the reduced Euler characteristic, *i.e.*, the Euler characteristic minus 1, of the absolute de Rham complex of $(V,0)$. In [57] is proved the following theorem:

Theorem 9.7.1. *For a curve singularity $(C,0)$,*

$$\nu(C,0) \;=\; \dim_{\mathbb{C}} \Omega^1_{C,0}/d\mathcal{O}_{C,0},$$

where d is the usual exterior derivative.

In other words this theorem says that the radial index (which is defined topologically) equals the difference between the Euler characteristics of the usual de Rham complex and the complex given by multiplication by the 1-form ω. This might be a special case of a general theorem for singular varieties in the spirit of the results of C. Simpson [152] and others for complex manifolds.

The idea of the proof is to consider the normalization $\pi : (\bar{C},\bar{0}) \to (C,0)$ of the curve and the commutative diagrams:

$$\begin{array}{ccccccccc}
0 \to & \mathcal{O}_{C,0} & \xrightarrow{\wedge\omega} & \Omega^1_{C,0} & \longrightarrow & \Omega^1_{C,0}/\omega \wedge \mathcal{O}_{C,0} & \to 0 \\
& \downarrow{\scriptstyle \pi_0^*} & & \downarrow{\scriptstyle \pi_1^*} & & \downarrow & \\
0 \to & \mathcal{O}_{\bar{C},\bar{0}} & \xrightarrow{\wedge\bar{\omega}} & \Omega^1_{\bar{C},\bar{0}} & \longrightarrow & \Omega^1_{\bar{C},\bar{0}}/\bar{\omega} \wedge \mathcal{O}_{\bar{C},\bar{0}} & \to 0
\end{array}$$

$$\begin{array}{ccccccc}
0 \to & m_{C,0} & \xrightarrow{d} & \Omega^1_{C,0} & \to & \Omega^1_{C,0}/d\mathcal{O}_{C,0} & \to 0 \\
& \downarrow{\scriptstyle \widetilde{\pi}_0^*} & & \downarrow{\scriptstyle \pi_1^*} & & \downarrow & \\
0 \to & m_{\bar{C},\bar{0}} & \xrightarrow{d} & \Omega^1_{\bar{C},\bar{0}} & \to & 0 &
\end{array}$$

where $m_{C,0}$ is the maximal ideal in the ring $\mathcal{O}_{C,0}$, $m_{\bar{C},\bar{0}}$ is the ideal of germs of functions on the normalization $(\bar{C}, \bar{0})$, equal to zero at all the points in $\bar{0}$. Then the snake Lemma yields to 9.7.1 (see [57, 4.3] for the complete proof).

Remark 9.7.1. A notion of a generalized Milnor number of a curve singularity $(C, 0)$ was introduced in [41] as $\dim_{\mathbb{C}} \omega_{C,0}/d\mathcal{O}_{C,0}$, where $\omega_{C,0}$ is the dualizing module of Grothendieck. For smoothable curve singularities, it is equal to $1 - \chi(\widetilde{C})$, where $\widetilde{C}$ is a smoothing of $(C, 0)$. Here we recall that all smoothings of a curve singularity have the same Euler characteristic. From the proof of Theorem 6.1.3 in [41], it follows that the Milnor number defined by R.-O. Buchweitz and G.-M. Greuel is equal to $\nu(C, 0) + \lambda - \tau$, where τ and λ are defined in Remark 9.6.3. For complete intersection curve singularities one has $\lambda = \tau$.

9.8 Indices for Collections of 1-Forms

We know already that on a smooth closed manifold M, the index of a vector field, or a 1-form, leads towards the Euler–Poincaré characteristic of M, which is the Poincaré dual of the top Chern class when the manifold is (almost) complex. In other words, if M has complex dimension m, then the total index of a 1-form ω on M with isolated singularities satisfies:

$$\mathrm{Ind}_{\mathrm{PH}}(\omega; M) \;=\; (-1)^m c^m(M)[M].$$

We also know from the previous chapters that one has similar statements for vector fields on compact complex analytic varieties, the precise statement one gets depending on the concept of index one is using. In fact, as we shall see in Chaps. 10–13 of this monograph, these are related to various concepts of "Chern classes" one has for singular varieties, which coincide with the usual Chern classes in the case of manifolds. For the radial index one gets $\chi(V)$, this is the 0-degree Schwartz–MacPherson class of V. For the GSV-index one gets the 0-degree Fulton–Johnson class of V, which (with some restrictions) equals the Euler–Poincaré characteristic of a smoothing $\widehat{V}$ as in the proof of

Theorem 3.2.2. For the local Euler obstruction one gets the top Chern class of the Nash bundle over the Nash blow up of V.

Now, in the case of manifolds the number $c^m(M)[M]$ is one of the Chern numbers the manifold has, but there are several others. One has a Chern number

$$c^{j_1}(M)\, c^{j_2}(M) \,\cdots c^{j_r}(M)\,[M],$$

whenever $j_1, ..., j_r$ are positive integers which add up to m. In the next chapters of this monograph we shall explore various ways of generalizing the Chern classes of complex manifolds to the case of singular varieties, and as we shall explain, these are related in one or another way to studying indices of appropriate vector fields or frames, as described in Chap. 1. These yield to homology (or cohomology) classes which represent various generalizations of Chern classes for singular varieties (of course there can be other means to constructing Chern classes for singular varieties, for instance using the MacPherson functor [117]).

Yet, there is another question that arises naturally in the context of this book: the $c^{j_1}(M)$ are cohomology classes in $H^{2j_i}(M;\mathbb{Z})$, but the evaluation $c^{j_1}(M)c^{j_2}(M)\cdots c^{j_r}(M)[M]$ is an actual number, in fact an integer. Is there a way of defining *an index* associated to this number in a similar way as the local Poincaré–Hopf index of a vector field (or 1-form) is associated to the Chern number $c^m(M)[M]$? Moreover, what can we say about this question when the ambient space is now a singular variety? what information these invariants give about singular varieties? These and other questions are addressed by W. Ebeling and S. Gusein-Zade in a series of articles (see [53–56]).

Before looking at this matter, let us envisage some related facts about the Chern numbers of manifolds.

We recall from Chap. 1 that given a complex manifold M of dimension m, its Chern class $c^r(M) \in H^{2r}(M)$ is the primary obstruction to constructing an $(m-r+1)$-frame in M. In other words, let $W_{m-r+1}(m)$ be the Stiefel manifold of complex orthonormal $(m-r+1)$-frames in $\mathbb{C}^m$. This manifold is diffeomorphic to $U(m)/U(r-1)$ and therefore it is $(2r-2)$-connected and its first nonzero homology and homotopy groups are $H^{2r-1}(W_{m-r+1}(m)) \cong \pi_{2r-1}(W_{m-r+1}(m)) \cong \mathbb{Z}$ (see [153]).

Now let $W_{m-r+1}(m)(TM)$ denote the fibre bundle over M whose fibre at each point x is the Stiefel manifold $W_{m-r+1}(m)$ of complex orthonormal $(m-r+1)$-frames in $T_xM \cong \mathbb{C}^m$. Let us try to construct a section of this bundle via the usual stepwise process. We triangulate M in some (any) way and construct a section of $W_{m-r+1}(m)(TM)$ step by step, starting from the 0-skeleton, then the 1-skeleton and so on, as far as we can. The fact that the fiber is $(2r-2)$-connected tells us that we can construct such a section up to the $(2r-1)$-skeleton of the triangulation. The first possibly nonzero obstruction arises when we try to extend the section over the $2r$-skeleton. We thus get an element in $\pi_{2r-1}(W_{m-r+1}(m)) \cong \mathbb{Z}$ associated to each $2r$-cell (or simplex). This defines a cochain of dimension $2r$, which is actually a cocycle and, by definition, represents the cohomology class $c^r(M)$. Notice that the Poincaré dual of $c^r(M)$ is a homology class of dimension $2m - 2r$.

Now look at the class $c^{m-r}(M)$. The analogous discussion says that this is the primary obstruction to constructing an $r+1$-frame on M. This class lives in $H^{2m-2r}(M)$ and its Poincaré dual is a homology class of dimension $2r$.

Suppose we can represent the Poincaré dual of $c^r(M)$ by an oriented submanifold C_{m-r} of M, which therefore has dimension $2m-2r$. Similarly, let C_r be an oriented submanifold of M of dimension $2r$ representing the Poincaré dual of $c^{m-r}(M)$. If these two manifolds intersect, then we can always move them slightly to make them have transverse intersections. By dimensional reasons, this means that they meet at points, which come equipped with a ± 1, depending on whether or not the tangent spaces of C_r and C_{m-r} at the meeting point yield the positive or negative orientation of T_xM. Counting all these points with their corresponding signs we get an integer that we may denote $C_{m-r} \cdot C_r$. This integer is precisely the Chern number $c^r(M)\, c^{m-r}(M)[M]$.

Geometrically this means that away from C_{m-r} we have a frame $v_1^{(r)} = (v_1^1, ..., v_{r+1}^1)$, and these vector fields become linearly dependant when we are in C_r. Similarly, away from C_{m-r} we have a frame $v_2^{(n-r)} = (v_1^2, ..., v_{m-r+1}^2)$, and the vector fields in this frame become linearly dependant when we are in C_r. The points that contribute towards the Chern number $c^r(M)\, c^{m-r}(M)[M]$ are those in the intersection of C_r and C_{m-r}. This inspires the following definition (which mimics that in [55]).

A point $x \in M$ is nonsingular for the collection of vector fields

$$\{(v_1^1, ..., v_{r+1}^1), (v_1^2, ..., v_{m-r+1}^2)\}$$

if at least one of the two sets of vectors $\{v_1^1, ..., v_{r+1}^1\}$ and $\{v_1^2, ..., v_{m-r+1}^2)\}$ is linearly independent at x. Otherwise we say that x is a singular point of the collection $\{v_1^{(r)}, v_2^{(n-r)}\}$.

In other words, the singular points of the collection of vector fields

$$\{(v_1^1, ..., v_{r+1}^1), (v_1^2, ..., v_{m-r+1}^2)\}$$

are the points that count for the Chern number $c^r c^{m-r}[M]$. And the way each singular point contributes towards this Chern number in the example above is ± 1 because of the transversality assumptions we made. In general this is an integer that can be regarded as a local index associated to the corresponding collection of vector fields at each singular point.

Now suppose we are given integers $r_1, ..., r_s$ such that $r_1 + ... + r_s = m$. One has a Chern number $c^{r_1}(M) \cdots c^{r_s}(M)\,[M]$, and the previous discussion extends to this setting, to say that this number is the intersection number of the cycles representing the Poincaré duals of the corresponding Chern classes. Each Chern class c^{r_j} corresponds, by duality, to the set of points where a certain family of vector fields becomes linearly dependent, the singularities of the corresponding frame v^{m-r_j+1}. A point $x \in M$ *is nonsingular* for the collection of vector fields

$$\{v^{(m-r_1+1)}, \cdots, v^{(m-r_s+1)}\}$$

if at least one of the sets of vectors $v^{(m-r_j+1)}$ is linearly independent at x. Otherwise we say that x is *a singular point* of the collection of vector fields. Notice that each singular point x_0 of the collection comes naturally equipped with a *local index*,

$$\text{Ind}(\{v^{(m-r_1+1)}, \cdots, v^{(m-r_s+1)}\}, x_0) \in \mathbb{Z},$$

given by the intersection product of the cycles determined by the points where each of the sets of vectors $v^{(m-r_j+1)}$ fails to be linearly independent. More precisely, recall from Chap. 1 that the obstruction cocycle for a frame $v^{(m-r_j+1)}$ lives in dimension $2r_j$; such a frame can be assumed to be nonsingular over the $2r_j - 1$ skeleton of an appropriate cell decomposition of M, and it has at most isolated singularities in the $2r_j$-skeleton, located at the barycenter $\hat{\sigma}$ of each cell σ of dimension $2r_j$. At each such singular point, the frame has its local index $\text{Ind}(v^{(m-r_j+1)}, \hat{\sigma})$, as defined in 1.3.2. Then the local index of the collection at each singular point x_0 is the product of the local indices of the corresponding frames:

$$\begin{aligned}\text{Ind}(\{v^{(m-r_1+1)}, \cdots, v^{(m-r_s+1)}\}, x_0)\\ = \text{Ind}(v^{(m-r_1+1)}, x_0) \cdots \text{Ind}(v^{(m-r_1+1)}, x_0),\end{aligned}$$

recalling that a singularity of the collection means a singularity of each frame.

Now suppose M is a compact, almost complex manifold with nonempty boundary ∂M, and we are given a collection of vector fields

$$\mathcal{V} = \{v^{(m-r_1+1)}, \cdots, v^{(m-r_s+1)}\},$$

on a neighborhood U of ∂M in M and with no singularities of the collection U. Then classical obstruction theory, as explained in Chap. 1, says that we can extend this collection of vector fields to all of M with finitely many singularities, and their total sum, counted with their local index, is independent of the extension. We thus have a Chern number $(c^{r_1} \cdots c^{r_s})_{\mathcal{V}}(M)[M, \partial M] \in \mathbb{Z}$ which depends only on M and the choice of the collection $\mathcal{V}$ near the boundary. We may call this *the Chern number of M relative to the collection $\mathcal{V}$*, in analogy with the relative Chern classes introduced in Chap. 1.

Notice that similar considerations apply if we replace TM by some other complex bundle over M of same dimension, in particular the cotangent bundle T^*M. Furthermore, we can make similar considerations for other complex vector bundles of dimension m over complex varieties of dimension m, as for instance the Nash bundle over the Nash blow up of a singular variety, and we shall do so in a moment.

These ideas are all behind the work of Ebeling and Gusein-Zade about indices of collections of 1-forms on singular varieties, that we now envisage.

There are two main situations we consider, following [53–56]. One of this leads to a generalization of the GSV-index, the other to a generalization of the Euler defect.

9.8.1 The GSV Index for Collections of 1-Forms

Consider an ICIS germ $(V,0)$ defined by a holomorphic map $f : (\mathbb{C}^{n+k},0) \to (\mathbb{C}^k,0)$, and let $\mathcal{V}^*$ be a collection of 1-forms on V with an isolated singularity at 0. That is, $\mathcal{V}^*$ consists of a set $\{\omega^{(n-r_1+1)}, \cdots, \omega^{(n-r_s+1)}\}$ of frames of 1-forms on $V^* := V \setminus \{0\}$, each such frame consisting of a number r_j of linearly independent 1-forms on V, linearly independent in some neighborhood of 0 in V. For simplicity we assume, with no loss of generality, that the representative of V is small enough, so that each frame $\omega^{(n-r_1+1)}$ is nonsingular on all of $V^* := V \setminus \{0\}$.

Let $\varepsilon > 0$ be small enough so that $K = V \cap \mathbb{S}_\varepsilon$ is the link of V, and let $0 < \delta << \varepsilon$ be small enough so that for each regular value t of f with $|t| \leq \delta$ one has that the fiber $F_t = f^{-1}(t)$ meets the sphere $\mathbb{S}_\varepsilon$ transversally. As in Chap. 3, we have that F_t is an almost complex manifold with boundary, and the transversal Isotopy Lemma (or the first Thom–Mather Isotopy theorem) tell us that the boundary ∂F_t is isotopic (in the ambient space) to K. Therefore, we can move the collection $\mathcal{V}^*$ to a collection of $1-forms$ on a neighborhood of ∂F_t in F_t, with no singular point. By the previous discussion, this collection determines a well-defined Chern number of the cotangent bundle T^*F_t relative to the collection of 1-forms $\mathcal{V}^*$.

This is the GSV index of the collection of 1-forms defined by Ebeling and Gusein-Zade. If the collection $\mathcal{V}^*$ consists of a single 1-form, this is the GSV index previously envisaged in this chapter. The same construction applied to a vector field is the GSV index of Chap. 3.

In concordance with Theorem 9.5.1, if all the 1-forms in the collection are holomorphic, then the authors express this index as the dimension of a certain vector space (see [55, Theorem 2.2]). This generalizes the Lê-Greuel formula for the Milnor number of an ICIS germ.

9.8.2 Local Chern Obstructions

Assume now that $(V,0)$ is the germ of a reduced complex analytic space in $\mathbb{C}^m$ of pure dimension n, and equip V with a Whitney stratification so that $\{0\}$ is a stratum. If ω is a 1-form on $\mathbb{C}^m$ with an isolated singularity at 0, then one has its local Euler obstruction defined in 9.3.1 above. We recall briefly its definition. Let $\widetilde{V} \xrightarrow{\nu} V$ be the Nash blow up of V, and $\widetilde{T}^* \xrightarrow{\pi} \widetilde{V}$ the dual of its Nash bundle. Then the 1-form ω lifts canonically to a section $\widetilde{\omega}$ of $\widetilde{T}^*$ over

$\nu^{-1}((V \cap \mathbb{B}_\varepsilon) \setminus \{0\})$, where $\mathbb{B}_\varepsilon$ is a small ball centered at 0. Its local Euler obstruction, denoted $\mathrm{Eu}_{\omega,\mathrm{V}}(0)$, is by definition the obstruction to extending $\widetilde{\omega}$ as a section of $\widetilde{T}^*$ over $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$. More precisely, if we denote such an obstruction by $\mathrm{Obs}(\widetilde{\omega}, \nu^{-1}(V \cap \mathbb{B}_\varepsilon)) \in H^{2n}(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \partial\mathbb{B}_\varepsilon))$, then by definition $\mathrm{Eu}_{\omega,\mathrm{V}}(0)$ is the integer one gets by evaluating $\mathrm{Obs}(\widetilde{\omega}, \nu^{-1}(V \cap \mathbb{B}_\varepsilon))$ in the fundamental cycle of the pair $(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \partial\mathbb{B}_\varepsilon))$.

This corresponds to considering the top Chern class of the Nash bundle over the Nash blow up, relative to the 1-form defined on the boundary $\nu^{-1}(V \cap \partial\mathbb{B}_\varepsilon)$.

Essentially the same construction goes through for collections of 1-forms on V instead of a single 1-form, but in this case one must pay attention not only to the singularities of the collection, but also to another type of "bad points," called *special points*. One gets the *local Chern obstructions* of Ebeling and Gusein-Zade. This defines local invariants of the germ $(V, 0)$ that generalize the local Euler obstruction of 1-forms.

Furthermore, one has that for an ICIS germ (V, P), the difference between the GSV index and the local Chern obstruction of a collection of 1-forms does not depend on the collection, so it is an invariant of the singularity (Theorem 3.4 in [55]). It would be interesting to explore these new invariants of singularities.

We refer to [52–56] for more on this interesting topic.

Chapter 10
The Schwartz Classes

Abstract As mentioned before, the first generalization of Chern classes to singular varieties is due to M.-H. Schwartz, using obstruction theory and radial frames. These classes are the primary obstructions to constructing a special type of stratified frames on V that she called radial frames. To avoid possible misunderstandings, here we prefer to call them *frames constructed by radial extension*, as in the case of vector fields. We refer to [28, 33] for details of the construction and we content ourselves with summarizing here their main properties. It was shown in [33] that these classes correspond, by Alexander isomorphism, to the MacPherson classes, that we discuss briefly in the last section of this chapter.

In this chapter we provide a viewpoint for studying Schwartz–MacPherson classes which is particularly close to the theory of indices of vector fields that we develop in this book, both from the topological and the differential geometric sides. In the first three sections, we discuss the Schwartz index of frames and a method for defining the Schwartz classes of singular varieties using arbitrary stratified frames, not necessarily constructed by radial extension. As a corollary we obtain that the Schwartz classes are the primary obstruction to constructing a stratified frame (any frame, not necessarily radial) on the skeleton of the appropriate dimension and for an appropriate cellular decomposition: if such a frame exists, then the corresponding Schwartz class vanishes (the converse is false in general).

In Sect. 4, we use the methods of [31], joint work with D. Lehmann, for constructing localized Schwartz classes in both the topological and differential geometric contexts, via Chern–Weil theory and using stratified frames. The last section discusses briefly MacPherson and Mather classes (see [28, 117]).

10.1 The Local Schwartz Index of a Frame

We know already from Chap. 2 how to define the Schwartz index of a vector field constructed by radial extension, and that notion was generalized to give an index for vector fields in general, using the difference between the given

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vector field and a radial one (2.4.2). In this section we extend these concepts to frames, using the *difference cocycle* (see [153]), in order to define the local Schwartz index for arbitrary stratified frames.

The idea for constructing frames by radial extension is similar to that for constructing vector fields by radial extension, that we described in Chap. 2 (see [142]). We consider as before, a compact, complex analytic n-dimensional variety V embedded in a complex m-manifold M, endowed with a Whitney stratification $\{V_\alpha\}$ adapted to V. We use a cellular decomposition (D), dual to a triangulation of M compatible with the stratification. The cells σ of (D) are transverse to the strata V_α. In general, elements $\sigma \cap V_\alpha$ of (D) are not cells, but that is the case for the smallest dimensional stratum V_α meeting σ.

The concept of stratified vector fields, introduced in Chap. 2, extends in the obvious way to frames:

Definition 10.1.1. Let A be a subspace of M. A *stratified r-frame* on A is an r-field $v^{(r)} = \{v_1, ..., v_r\}$ consisting of stratified vector fields $v_1, \dots, v_r$, linearly independent everywhere. By a *singularity* of an r-frame we mean a point $z \in A$ where the r vectors $v_1(z), \dots, v_r(z)$ fail to be linearly independent.

Let σ be a (D)-cell of dimension $2(m - r + 1)$, dual of a simplex in the (complex) d-dimensional stratum V_α. Then $\sigma_\alpha = \sigma \cap V_\alpha$ is a cell of (real) dimension $2d - 2r + 2$. Suppose we have a stratified r-frame $v^{(r)}$ defined on the boundary of σ_α. We thus have an associated map $v^{(r)} : \partial\sigma_\alpha \to W_{r,d}$, which determines an element in $\pi_{2d-2r+1}(W_{r,d}) \simeq \mathbb{Z}$. *This defines an index* $\mathrm{Ind}(v^{(r)}, \sigma_\alpha) \in \mathbb{Z}$, as in Chap. 1, that by abuse of notation we call *the Poincaré–Hopf index* of the frame $v^{(r)}$ on σ_α. The frame can of course be extended by a homothecy to all of σ_α minus its barycenter a_σ, Notice that if we write the frame $v^{(r)}$ as $(v^{(r-1)}, v_r)$, where $v^{(r-1)}$ is the $(r-1)$ frame determined by the first $r-1$ vector fields in $v^{(r)}$, then $v^{(r-1)}$ extends without singularities to the interior of σ_α, by dimensional reasons. It spans a bundle $\mathrm{Sp}\{v^{(r-1)}\}$. Then the index $\mathrm{Ind}(v^{(r)}, \sigma_\alpha)$ equals the degree of the map from $\partial\sigma_\alpha$ into the unit sphere of the fiber over a_σ of orthogonal complement of the bundle $\mathrm{Sp}\{v^{(r-1)}\}$, defined by the last vector field v_r (normalized).

The M.-H. Schwartz's radial extension technique can be used to extend each of the components of $v^{(r)}$ in a neighborhood of σ_α in M (see [28]). In this way we obtain a stratified r-frame on a punctured neighborhood of a_σ in the $2(m - r + 1)$-cell σ of (D). The boundary of σ is a $(2m - 2r + 1)$-sphere. This defines a map $\partial\sigma \to W_{r,m}$ which represents an element in $\pi_{2m-2r+1}(W_{r,m}) \simeq \mathbb{Z}$ and *therefore defines an index* in $\mathbb{Z}$. Since by construction the frame is radial in all directions normal to the stratum V_α, it follows that this index coincides with the previously defined index $\mathrm{Ind}(v^{(r)}, \sigma_\alpha)$. One has:

Theorem 10.1.1. *Let V_α be a (complex) d-dimensional stratum in V and suppose $\sigma_\alpha = \sigma \cap V_\alpha$ is a $(2d-2r+2)$-cell. Let $v^{(r)}$ be an r-frame defined on the boundary of σ_α Let us write $v^{(r)} = (v^{(r-1)}, v_r)$ where $v^{(r-1)}$ is the $(r-1)$ frame determined by the first $r-1$ components (vector fields) in $v^{(r)}$. Then:*

(1) *the* $(r-1)$*-frame* $v^{(r-1)}$ *can be extended to a frame (with no singularities) on the whole cell* σ *and*

(2) *the index* $\mathrm{Ind}_{\mathrm{PH}}(v^{(r)},\sigma)$ *of* $v^{(r)}$ *in* σ *can be defined in either of the three equivalent ways:*

(a) *as the element in* $\pi_{2d-2r+1}(W_{r,d}) \simeq \mathbb{Z}$ *represented by* $\partial\sigma_\alpha \xrightarrow{v^{(r)}} W_{r,d}$,

(b) *as the element in* $\pi_{2d-2r+1}(\mathbb{S}^{2d-2r+1}) \simeq \mathbb{Z}$ *represented by* $\partial\sigma_\alpha \xrightarrow{v_r} \mathbb{S}^{2d-2r+1}$*; the map into the fiber of the orthogonal complement of the bundle* $Sp\{v^{(r-1)}\}$ *spanned by* $v^{(r-1)}$ *and*

(c) *as the element in* $\pi_{2m-2r+1}(W_{r,m}) \simeq \mathbb{Z}$ *represented by* $\partial\sigma \xrightarrow{\widehat{v}^{(r)}} W_{r,m}$, *where* $\widehat{v}^{(r)}$ *is a (stratified) radial extension of* $v^{(r)}$ *to the boundary of the cell* $\sigma \in (D)$ *whose intersection with* V_α *is* σ_α.

Definition 10.1.2. Let $v^{(r)}$ be an r-frame as above, defined on the boundary of a cell $\sigma \cap V_\alpha$ of dimension $2d-2r+2$, where d is the dimension of the stratum V_α. The integer obtained in 10.1.1 is *the Schwartz index* of the r-frame $v^{(r)}$ on the cell σ. We denote it $\mathrm{Ind}_{\mathrm{Sch}}(v^{(r)},\sigma)$.

Example 10.1.1. Consider the 2-cell e^2 in $\mathbb{C}^3$ defined by $\{(0,0,z) \big| |z| \leq 1\}$. The 3-frame on e^2 defined by $v^{(3)} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ obviously has index 0, since it does not vanish anywhere. Consider now the 3-frame on e^2 defined by:

$$v^{(3)} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, z^k \frac{\partial}{\partial z}), \qquad k \geq 1.$$

Using the second definition above we see that the index of this frame equals the Poincaré–Hopf index at $0 \in \mathbb{C}$ of the holomorphic vector field $z^k \frac{\partial}{\partial z}$, so it has index k. Notice that we may consider only a 2-frame on e^2, for example $v^{(2)} = (\frac{\partial}{\partial x}, z^k \frac{\partial}{\partial z})$; this has a singularity at $\{(0,0,0)\}$, but in this case one can get rid of this singularity by deforming the frame by an appropriate homotopy, since the 2-frame on the boundary extends to the interior if and only if the corresponding map into $W_{2,3}$ is nulhomotopic, but this Stiefel manifold is diffeomorphic to $\mathbb{S}^3$, so it is simply connected. In the previous case the 3-frames on $\partial e^2 = \mathbb{S}^1$ correspond to the elements in the fundamental group of the unitary group $U(3) \simeq W_{3,3}$, $\pi_1(U(3)) \simeq \mathbb{Z}$.

Let σ be a cell of (D) of dimension $2m-2r+2$ that meets the (complex) d-dimensional stratum V_α along the cell σ_α. One denotes by a_σ the barycenter of the cell σ. Let the stratified frame $v^{(r)}$ be defined (with no singularity) on the boundary of σ. One defines:

Definition 10.1.3. We say that $v^{(r)}$ is *normally radial* at a_σ if for each stratum V_β having a_σ in its closure and for each sufficiently small tube $\mathcal{T}_\varepsilon(V_\alpha)$ around V_α in M, one has that each component $v_1, ..., v_r$ of $v^{(r)}$ is transverse (pointing outwards) to the intersection $\overline{V}_\beta \cap \mathcal{T}_\varepsilon(V_\alpha)$.

As in the case of vector fields, up to homotopy realized by stratified vector fields, every normally radial frame at a_σ is obtained by radial extension of its restriction to V_α.

For normally radial frames one has that its Poincaré–Hopf index in the stratum equals the Poincaré–Hopf index in M, and so one has a well defined Schwartz index as before. Notice this says nothing about the behavior of the frame in the stratum V_α containing a_σ.

Example 10.1.2. Consider the 4-cell $e = \{(0,y,z) \big| |y^2| + |z|^2 \leq 1\}$ in $\mathbb{C}^3 = \{(x,y,z)\}$ and the Whitney stratification of the unit ball $\mathbb{B}$ in $\mathbb{C}^3$ consisting of the strata $\big\{e, \mathbb{B} \setminus \{e\}\big\}$. The 2-frame

$$v^{(2)} = (x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y},\, x\,\frac{\partial}{\partial x} + z^k\,\frac{\partial}{\partial z}), \qquad k \geq 1,$$

is stratified, normally radial and it has Schwartz index k.

Now we wish to distinguish the normally radial stratified frames with the "simplest" possible behavior in the stratum V_α of a_σ. Write such a frame $v^{(r)}$ in the form $(v^{(r-1)}, v_r)$ as before; we assume the frame $v^{(r-1)}$ is extended to the cell σ with no singularity, so the only singularity of $v^{(r)}$ in σ is that of the vector field v_r at the barycenter a_σ. Let $Q \subset TM|_{\tilde{\sigma}}$ be the orthogonal complement (for some Riemannian metric) of the bundle $\mathrm{Sp}\{v^{(r-1)}\}$; the Schwartz index of $v^{(r)}$ at a_σ is the Poincaré–Hopf index of the section v_r of Q.

Definition 10.1.4. Let $v^{(r)}$ be a stratified r-frame defined on a neighborhood in M of a point x in a stratum V_α of V of dimension d. We say that $v^{(r)}$ is *radial* at x if it is normally radial in M and it is radial in the stratum of x, *i.e.*, it has Schwartz index 1.

Example 10.1.3. With the same data than Example 10.1.2, the 2-frame $v^{(2)} = \big(\frac{\partial}{\partial y},\, z\,\frac{\partial}{\partial z}\big)$ is radial on the stratum e of $\mathbb{B}$, but it is not normally radial. The frame

$$\Big(x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y},\, x\,\frac{\partial}{\partial x} + z\,\frac{\partial}{\partial z}\Big),$$

is radial in the stratum e and it is also normally radial, so it is radial in the ambient space.

We now define the local Schwartz index for arbitrary (stratified) frames; this is similar to 2.4.2. Let $v^{(r)}$ be an r-frame defined on the boundary of a (D)-cell σ of dimension $2m - 2r + 2$, whose barycenter is a point $a_\sigma \in V_\alpha \subset V$. We extend $v^{(r)}$ to a stratified frame on all of $\sigma \setminus \{a_\sigma\}$. Recall that, by construction, the cell σ meets transversally all the Whitney strata V_β containing V_α in their closure. Let $v^{(r)}_{\mathrm{rad}}$ be a stratified radial frame around a_σ. We define the difference between $v^{(r)}$ and $v^{(r)}_{\mathrm{rad}}$ at a_σ as follows. Consider sufficiently small spheres $\mathbb{S}_\varepsilon$, $\mathbb{S}_{\varepsilon'}$ in M, $\varepsilon > \varepsilon' > 0$, centered at a_σ, and consider the frame $v^{(r)}$ on $\mathbb{S}_\varepsilon \cap \sigma \cap V$ and $v^{(r)}_{\rm rad}$ on $\mathbb{S}_{\varepsilon'} \cap \sigma \cap V$. We use again the Schwartz's technique of radial extension to get a stratified r-frame $w^{(r)}$ on the intersection of σ with the cylinder

$$X = [(V \cap \mathbb{B}_e) \setminus (V \cap \overset{\circ}{\mathbb{B}}_{e'})]$$

in V bounded by $\mathbf{K}_\varepsilon = \mathbb{S}_\varepsilon \cap V$ and $\mathbf{K}_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap V$, having finitely many singularities in the interior of X. At each of these singular points its index in the stratum, $\mathrm{Ind}_{\rm PH}(w^{(r)}, X \cap \sigma)$, equals its index in the ambient space $\mathbb{C}^m$. The *difference* of $v^{(r)}$ and $v^{(r)}_{\rm rad}$ is defined as:

$$d(v^{(r)}, v^{(r)}_{\rm rad}) = \sum \mathrm{Ind}_{\rm PH}(w^{(r)}, X \cap \sigma),$$

where the sum on the right runs over the singular points of $w^{(r)}$ in X and each singularity is being counted with the local index of $w^{(r)}$ in the corresponding stratum. As in the work of M.-H. Schwartz, we can check that this integer does not depend on the choice of $w^{(r)}$.

Definition 10.1.5. *The Schwartz* (or radial) *index* of the stratified r-field $v^{(r)}$ at $a_\sigma \in V$ is:

$$\mathrm{Ind}_{\rm Sch}(v^{(r)}, a_\sigma; V) \;=\; 1 + d(v^{(r)}, v^{(r)}_{\rm rad}) \;=\; 1 + \sum \mathrm{Ind}(w^{(r)}, X \cap \sigma),$$

where the sum on the right is taken over the singularities of $w^{(r)}$ in all strata in $X \cap \sigma$.

10.2 Proportionality Theorem

The Proportionality Theorem, due to [33], is the key point in the proof of the equality of Schwartz and MacPherson classes via the Alexander isomorphism.

Let $\nu : \widetilde{V} \to V$ be the Nash modification of V. Let σ be a cell of dimension $2(m-r+1)$ and $v^{(r)}$ a stratified r-field on $\sigma \cap V$ with an isolated singularity at the barycenter a_σ of σ. Since $v^{(r)}$ is nonsingular on $\partial\sigma \cap V$, it can be lifted to an r-frame $\widetilde{v}^{(r)}$ of $\widetilde{T}$ over $\nu^{-1}(\partial\sigma \cap V)$, as in the case of vector fields. One obtains, over $\nu^{-1}(\partial\sigma \cap V)$, a section of the bundle $\widetilde{T}_r$ associated to $\widetilde{T}$ and whose fiber in a point $\tilde{x}$ is the set of r-frames in $\widetilde{T}_{\tilde{x}}$. Let $o(\widetilde{v}^{(r)})$ denote the class in $H^{2(n-r+1)}(\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V))$ of the obstruction cocycle to extending $\widetilde{v}^{(r)}$ further to a section of $\widetilde{T}_r|_{\nu^{-1}(\widetilde{\sigma})}$.

Definition 10.2.1. The *local Euler obstruction* $\mathrm{Eu}_V(v^{(r)}, a_\sigma)$ of the stratified r-field $v^{(r)}$ at the isolated singularity a_σ is the integer obtained by evaluating $o(\widetilde{v}^{(r)})$ on the orientation cycle $[\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)]$.

Theorem 10.2.1. [33] *Let $v^{(r)}$ be a stratified r-frame on $\sigma \setminus \{a_\sigma\}$, where σ is a cell of (D) of dimension $2m - 2r + 2$ and a_σ is its barycenter. Let V_α be the stratum that contains a_σ, $\dim_{\mathbb{C}} V_\alpha = d$. Assume further that $v^{(r)}$ is normally radial at a_σ. One has:*

$$\mathrm{Eu}_V(v^{(r)}, a_\sigma) = \mathrm{Eu}_V(a_\sigma) \cdot \mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}, \sigma).$$

A possible proof is to use *the stability of the Euler obstruction* $\mathrm{Eu}_V(v^{(r)}, a)$ *under appropriate perturbations* and to prove the Theorem in the same way as Theorem 3.6.1. We give a proof adapted from the one of [33] but which is simpler (see [35]).

Proof. If $r = 1$, this is Theorem 8.1.2.

If $r > 1$, we know that, up to homotopy, the normally radial frame can be considered as obtained by radial extension of its restriction to V_α. Then, we reduce the problem to the case $r = 1$ in the following way.

1. Taking, if necessary, a smaller cell σ (that is necessary if the boundary of $\sigma \cap V_\alpha$ does not lie entirely in the interior of the stratum V_α), one proceeds as follows: Let us denote by $\sigma_\alpha = \sigma \cap V_\alpha$, and consider $\mathbb{D}$ a small closed disc of complex dimension $r-1$ with center p transverse to σ_α in V_α. By the local triviality property of Whitney stratifications, there is a neighborhood Ω of p in M homeomorphic to $\mathbb{D} \times \sigma$ and stratified by $\{\mathbb{D} \times (\sigma \cap V_\beta)\}_\beta$ where V_β are the strata which meet σ.

Let us write $v_\alpha^{(r)}$ as $(v_\alpha^{(r-1)}, v_{\alpha,r})$, where $v_\alpha^{(r-1)}$ is the $(r-1)$-frame consisting of the first $(r-1)$ vector fields. By hypothesis, $v_\alpha^{(r-1)}$ has no singularity on $\partial\sigma \cap V_\alpha$; and by dimensional reasons, obstruction theory says that it extends to a vector field on all of $\sigma \cap V_\alpha$ (because it represents a class in a homotopy group which is trivial). Thus we may choose a radial extension $v^{(r)} = (v^{(r-1)}, v_r)$ on σ where the $(r-1)$-field $v^{(r-1)}$ is nonsingular and v_r is a radial extension of the vector field $v_{\alpha,r}$.

Let E denote the trivial subbundle of $TM|_\sigma$ of rank $r-1$ spanned by $v^{(r-1)}$ (over the complex numbers) and Q the orthogonal complement of E in $TM|_\sigma$ for some metric;

$$TM|_\sigma = E \oplus Q. \tag{10.2.2}$$

The bundle E extends to a neighborhood of p in M and we get, denoting also by E and Q the extensions of E and Q, a decomposition

$$TM|_\Omega = E \oplus Q. \tag{10.2.3}$$

The bundles E and Q can be interpreted as $\pi_1^* T\mathbb{D}$ and $\pi_2^* T\sigma$, respectively, where $\pi_1 : \Omega \to \mathbb{D}$ and $\pi_2 : \Omega \to \sigma$ denote the projections.

Now we may think of v_r as a stratified section of $T\sigma$. Let $v_{\mathbb{D}}$ denote a radial vector field on $\mathbb{D}$ at p. Then the sum

$$v_0 = \pi_1^* v_{\mathbb{D}} + \pi_2^* v_r \tag{10.2.4}$$

is a vector field on Ω, singular at p. We claim that v_0 is stratified ; that follows from the choice of the stratification in $\Omega = \mathbb{D} \times \sigma$, the fact that v_r is a stratified vector field on σ and the fact that $\pi_1^* v_{\mathbb{D}}$ or $\pi_2^* v_r$ is tangent to each fiber of π_2 or π_1, respectively. Moreover, from the construction we see that v_0 is a radial extension of the vector field $(\pi_1|_{V_\alpha})^* v_{\mathbb{D}} + (\pi_2|_{V_\alpha})^* v_{\alpha,r}$ on V_α.

Let $\widetilde{v}_0$ be the lifting of v_0 as a section of $\widetilde{T}$ over $\nu^{-1}(\mathbb{S} \cap V)$, where $\mathbb{S} = \partial\Omega$. If we denote by $o(\widetilde{v}_0, \widetilde{T})$ the class of the obstruction cocycle to extending $\widetilde{v}_0$ to a nonvanishing section over $\nu^{-1}(\Omega \cap V)$, by definition we have

$$\mathrm{Eu}(v_0, V; p) = o(\widetilde{v}_0, \widetilde{T})[\nu^{-1}(\Omega \cap V), \nu^{-1}(\mathbb{S} \cap V)].$$

In the following steps 2 and 3, we prove that

$$\mathrm{Eu}(v^{(r)}, V; p) = \mathrm{Eu}(v_0, V; p). \tag{10.2.5}$$

This will finish the proof of Theorem 10.2.1, since Theorem 8.1.2 implies

$$\mathrm{Eu}(v_0, V; p) = \mathrm{Eu}(V, p) \cdot \mathrm{Ind}_{\mathrm{Sch}}(v_0, V; p)$$

and by definition, we have

$$\mathrm{Ind}_{\mathrm{Sch}}(v_0, V; p) = \mathrm{Ind}_{\mathrm{Sch}}(v_r, V; p) = \mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}, V; p).$$

2. Expression of $\mathrm{Eu}(v^{(r)}, V; p)$.

Using the decomposition (10.2.2), we obtain a decomposition on $\nu^{-1}(\sigma)$:

$$\nu^* TM|_\sigma = \nu^* E \oplus \nu^* Q.$$

Since the r-field $(v^{(r-1)}, v_r)$ is stratified and nonsingular on $\partial\sigma \cap V$, it lifts to an r-frame $\widetilde{v}^{(r)} = (\widetilde{v}^{(r-1)}, \widetilde{v}_r)$ of the Nash bundle $\widetilde{T}$ over $\nu^{-1}(\partial\sigma \cap V)$. Moreover, since $v^{(r-1)}$ is nonsingular on $\sigma \cap V$, $\nu^* E$ (restricted to $\nu^{-1}(\sigma \cap V)$) is a subbundle of $\widetilde{T}|_{\nu^{-1}(\sigma\cap V)}$ and we have a decomposition:

$$\widetilde{T}|_{\nu^{-1}(\sigma\cap V)} = \nu^* E \oplus \widetilde{P}, \tag{10.2.6}$$

where $\widetilde{P}$ is a subbundle of $\nu^* Q|_{\nu^{-1}(\sigma\cap V)}$. We may think of $\widetilde{v}_r$ as a section of $\widetilde{P}$ which is nonvanishing on $\nu^{-1}(\partial\sigma \cap V)$. If we denote by $o(\widetilde{v}_r, \widetilde{P})$ the class in $H^{2(n-r+1)}(\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V))$ of the obstruction cocycle to extending this to a nonvanishing section over $\nu^{-1}(\sigma \cap V)$, we have

$$\mathrm{Eu}(v^{(r)}, V; p) = o(\widetilde{v}_r, \widetilde{P})[\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)]. \tag{10.2.7}$$

3. Expression of $\mathrm{Eu}(v_0, V; p)$.

Using the (stratified) decomposition (10.2.4), let $\widetilde{v}_0 = (\widetilde{\pi_1^* v_{\mathbb{D}}}, \widetilde{\pi_2^* v_r})$ be the lifting of the respective vector fields to sections of $\widetilde{T}$ over $\nu^{-1}(\mathbb{S} \cap V)$, where $\widetilde{\pi_1^* v_{\mathbb{D}}} = \nu^* \pi_1^* v_{\mathbb{D}}$ is a section of $\nu^* E$ and $\widetilde{\pi_2^* v_r}$ is a section of $\widetilde{P}$.

Now, in the decomposition (10.2.3), since E is in SV, *i.e.*, the vectors in E are stratified, the pull-back $\nu^* E$ (restricted to $\nu^{-1}(\Omega \cap V)$) is a subbundle of $\widetilde{T}|_{\nu^{-1}(\Omega \cap V)}$ and we have a decomposition

$$\widetilde{T}|_{\nu^{-1}(\Omega \cap V)} = \nu^* E \oplus \widetilde{P},$$

where $\widetilde{P}$ is a subbundle of $\nu^* Q|_{\nu^{-1}(\Omega \cap V)}$, extending $\widetilde{P}$ in (10.2.6).

Let us denote by $o(\nu^* \pi_1^* v_{\mathbb{D}}, \nu^* E)$ the class of the obstruction cocycle to extending $\nu^* \pi_1^* v_{\mathbb{D}}$, a section of $\nu^* E$ nonvanishing on $\nu^{-1}((\partial \mathbb{D} \times \sigma) \cap V) = \nu^{-1}(\partial \mathbb{D} \times (\sigma \cap V))$, to a nonvanishing section on $\nu^{-1}(\Omega \cap V)$. In the same way, we denote by $o(\widetilde{\pi_2^* v_r}, \widetilde{P})$ the class of the obstruction cocycle to extending $\widetilde{\pi_2^* v_r}$, a section of $\widetilde{P}$ nonvanishing on $\nu^{-1}((\mathbb{D} \times \partial\sigma) \cap V) = \nu^{-1}(\mathbb{D} \times (\partial\sigma \cap V))$, to a nonvanishing section on $\nu^{-1}(\Omega \cap V)$. Then we have

$$o(\widetilde{v}_0, \widetilde{T}) = o(\nu^* \pi_1^* v_{\mathbb{D}}, \nu^* E) \smile o(\widetilde{\pi_2^* v_r}, \widetilde{P}),$$

where $\smile$ denotes the cup product. We have $o(\nu^* \pi_1^* v_{\mathbb{D}}, \nu^* E) = \nu^* \pi_1^* o(v_{\mathbb{D}}, T\mathbb{D})$. Since $\Omega \cap V = \mathbb{D} \times (\sigma \cap V)$ and $o(v_{\mathbb{D}}, T\mathbb{D})$ is a generator of $H^{2r-2}(\mathbb{D}, \partial \mathbb{D})$, we get

$$o(\nu^* \pi_1^* v_{\mathbb{D}}, \nu^* E) \frown [\nu^{-1}(\Omega \cap V), \nu^{-1}(\mathbb{S} \cap V)] = [\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)],$$

where $\frown$ denotes the cap product. Since the restriction of $o(\widetilde{\pi_2^* v_r}, \widetilde{P})$ to $\nu^{-1}\sigma$ is equal to $o(\widetilde{v_r}, \widetilde{P})$, we obtain

$$\mathrm{Eu}(v_0, V; p) = o(\widetilde{v}_r, \widetilde{P})[\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)]. \tag{10.2.8}$$

From (10.2.8) and (10.2.7), we obtain (10.2.5) and the theorem.

Remark 10.2.1. Notice that both proofs above are easily adapted to prove the equivalent theorem for coframes on V, *i.e.*, for frames of 1-forms on V. This gives relations between the dual Schwartz classes of V (which can be represented by Chern classes of the cotangent bundle $T^*(U)$, U being now a regular neighborhood of V in M, relative to a coframe on $U \setminus V$ obtained by radial extension) and the corresponding Chern classes of the dual Nash bundle $\widetilde{T}^*$. This is related to the recent work of Ebeling and Gusein-Zade about indices of "collections" of 1-forms [55].

10.3 The Schwartz Classes

As we mentioned before, Schwartz classes are the obstructions to constructing stratified frames on V obtained by radial extension. We recall (see [28, 33] for details) that to constructing these frames we take appropriate stratifications, triangulations and dual cellular decompositions as before, and we construct r-frame v_{rad} with isolated singularities in the barycenters of the cells in $(D)^{(2m-2r+2)}$, by induction on the dimension of the strata. If, as before, we write frames as $v^{(r)} = (v^{(r-1)}, v_r)$, then one has that one can construct on the $2q$-skeleton $(D)^{(2q)}$ of (D) in M a frame $v^{(r)}_{\mathrm{rad}} = (v^{(r-1)}_{\mathrm{rad}}, v_{\mathrm{rad}})$ whose main properties are:

(1) $v^{(r)}_{\mathrm{rad}}$ has only isolated singularities on $(D)^{(2q)}$; these are the singularities of the last vector field v_{rad}, while the $(r-1)$ frame $(v^{(r-1)}_{\mathrm{rad}})$ is nonsingular on $(D)^{(2q)}$. The vector field v_{rad} is nonsingular on $(D)^{(2q-1)}$.

(2) it is everywhere pointing outwards from cellular tubes around V and cellular tubes around the strata V_α. That is, it is everywhere normally radial. Hence the Schwartz index $\mathrm{Ind}_{\mathrm{Sch}}$ can be evaluated as in either of the 3 ways described in Theorem 10.1.1 above.

These are the *frames obtained by radial extension*. A motivation for considering such frames is that they are in a sense “canonical,” up to homotopy, and these are the only frames one can construct “explicitly” on singular varieties in general.

Let us now denote by $\widehat{\mathcal{T}}_V$ a cellular tube around V in M and consider a radial r-frame $v^{(r)}_{\mathrm{rad}}$ on $(D)^{(2q)}$. By construction this frame is nonsingular on $\widehat{\mathcal{T}}_V \setminus V$. It determines a $2q$-cochain $\gamma^q \in C^{2q}(\widehat{\mathcal{T}}_V, \partial\widehat{\mathcal{T}}_V)$ whose value on a $2q$-cell σ is $< \gamma^q \cdot \sigma >= \mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}_{\mathrm{rad}}, \sigma)$ if σ intersects V and 0 elsewhere.

It is proved in [33, 141] that this cochain is actually a cocycle, representing a cohomology class $c^q(V) \in H^{2q}(\widehat{\mathcal{T}}_V, \partial\widehat{\mathcal{T}}_V) \simeq H^{2q}(M, M \setminus V)$. This class does not depend on the choices of the Whitney stratification of M, the triangulations, nor the r-frame $v^{(r)}$, so long as it is constructed by radial extension (see [141], [142] and [28]).

Definition 10.3.1. The class $c^q(V) \in H^{2q}(M, M \setminus V)$ is the q-th Schwartz class of V.

The same construction can be performed for a stratified $r - frame$, $r \geq 1$, which is nonsingular on $(D)^{(2m-2r+1)}$ and has isolated singularities on $(D)^{2m-2r+2}$. Then the Schwartz indices of $v^{(r)}$, defined as in 10.1.5, determine a cocycle in the same way. One obtains a relative class

$$c^q(\widehat{\mathcal{T}}_V, \partial\widehat{\mathcal{T}}_V; v^{(r)}) \in H^{2q}(\widehat{\mathcal{T}}_V, \widehat{\mathcal{T}}_V \setminus V) \cong H^{2q}(M, M \setminus V) \qquad (10.3.0)$$

and the following Theorem:

Theorem 10.3.1. *Given $V \subset M$ as before, equipped with a Whitney stratification adapted to V and a compatible triangulation (K), we let (D) be the dual cellular decomposition and denote $(D)^j$ the union of all cells of dimension j. If $v^{(r)}$ is a stratified $r-frame$, $r \geq 1$, which is nonsingular on $(D)^{(2m-2r+1)}$ and has isolated singularities on $(D)^{2m-2r+2}$, then the Schwartz indices of $v^{(r)}$, defined as in 10.1.5, determine a cocycle as in 10.3.1, $c^q(V; v^{(r)}) \in H^{2q}(M, M \setminus V)$, $q = 2m - 2r + 2$, and this cocycle represents the corresponding Schwartz class of V.*

The proof is immediate from the definitions and properties of Schwartz index. Just as for vector fields 2.4.1, one has:

Corollary 10.3.1. *With the above hypothesis and notation, if there exists some stratified $r-frame$ which is nonsingular on $(D)^{(2m-2r+2)}$, then the corresponding Schwartz class of V vanishes.*

Since $H^{2q}(M, M \setminus V) \simeq H_{2m-2q}(V)$ by Alexander duality, these classes can be thought of as living in the homology of V. We denote by $c_i(V)$, $i = m - q = n - p$, the image of $c^q(V)$ in $H_{2i}(V)$ and call it the homology Schwartz class.

10.4 Alexander and Other Homomorphisms

The basic reference for this section is [25] (see also [28] and [31]). Here we adapt to the singular case the discussion started in Sect. 1.3 concerning the Alexander homomorphism and other related topics.

Let V be a pure n-dimensional subvariety in an m-dimensional complex manifold M and let (K), (K') and (D) be as before, (K) is a triangulation of M adapted to V, (K') its first barycentric subdivision and (D) the corresponding dual cell decomposition.

First, if V is compact, we define a homomorphism

$$P : C^{2n-i}_{(K')}(V) \longrightarrow C^{(K)}_i(V) \qquad \text{by} \qquad P(c) = \sum_{\sigma} \langle c, d(s) \cap V \rangle \tag{10.4.1}$$

for a $(2n-i)$-cochain c and a $(2n+2k-i)$-cell $d(s)$ dual of the simplex s, where the sum is taken over all i-simplices σ of V. This induces a homomorphism

$$P_V : H^{2n-i}(V) \longrightarrow H_i(V),$$

which is called the Poincaré homomorphism. If V is nonsingular, this is the Poincaré isomorphism.

Next, let S be a compact (K)-subcomplex of V (V may not be compact). We define a homomorphism

$$A : C^{2n-i}_{(K')}(V, V \setminus S) \longrightarrow C^{(K)}_i(S)$$

taking, in the sum of (10.4.1), only i-simplices of S. Then this induces a homomorphism

$$A_{V,S} : H^{2n-i}(V, V \setminus S) \longrightarrow H_i(S),$$

which is called the Alexander homomorphism. If V is nonsingular, this is the Alexander isomorphism. If V is compact, we have a commutative diagram similar to the one in Sect. 1.2.

Next we introduce Gysin and Thom homomorphisms. We define a homomorphism

$$G : C^{2n-i}_{(K')}(V) \longrightarrow C^{2m-i}_{(D)}(M) \qquad \text{by} \qquad \langle G(c), d(s)\rangle = \langle c, d(s) \cap V\rangle \tag{10.4.2}$$

for a $(2n-i)$-cochain c and a $(2m-i)$-cell $d(s)$. This induces a homomorphism

$$G_{V,M} : H^{2n-i}(V) \longrightarrow H^{2m-i}(M), \tag{10.4.3}$$

which is called the Gysin homomorphism. From the definition, we see that, if M is compact, the following diagram is commutative:

$$\begin{array}{ccc} H^{2n-i}(V) & \xrightarrow{G_{V,M}} & H^{2m-i}(M) \\ \downarrow P_V & & \wr\downarrow P_M \\ H_i(V) & \xrightarrow{i_*} & H_i(M). \end{array} \tag{10.4.4}$$

The Gysin homomorphism $G_{V,M}$ is sometimes denoted by i_*.

Note that from the expression (10.4.2), we see that the homomorphism G lifts to a homomorphism

$$T : C^{2n-i}_{(K')}(V) \longrightarrow C^{2m-i}_{(D)}(M, M \setminus V). \tag{10.4.5}$$

This induces a homomorphism

$$T_{V,M} : H^{2n-i}(V) \longrightarrow H^{2m-i}(M, M \setminus V), \tag{10.4.6}$$

which is called the Thom homomorphism. From the definition, we see that, if V is compact, the following diagram is commutative:

$$\begin{array}{ccc} H^{2n-i}(V) & \xrightarrow{T_{V,M}} & H^{2m-i}(M, M \setminus V) \\ \downarrow P_V & & \wr\downarrow A_{M,V} \\ H_i(V) & \xrightarrow{=} & H_i(V). \end{array} \tag{10.4.7}$$

We may write $G_{V,M} = j^* \circ T_{V,M}$ with $j^* : H^{2m-i}(M, M \backslash V) \to H^{2m-i}(M)$ the canonical homomorphism.

Let S be a subset of V as before. In the homomorphism (10.4.2), if σ is a (D)-cell not intersecting with S, then $\sigma \cap V$ is a (K')-chain not intersecting with S. Thus it induces a homomorphism

$$T : C^{2n-i}_{(K')}(V, V \setminus S) \longrightarrow C^{2m-i}_{(D)}(M, M \setminus S). \tag{10.4.8}$$

This in turn induces a homomorphism

$$T_{S,V,M} : H^{2n-i}(V, V \setminus S) \longrightarrow H^{2m-i}(M, M \setminus S), \tag{10.4.9}$$

which is also called the Thom homomorphism. From the definition, we see that the following diagram is commutative:

$$\begin{array}{ccc} H^{2n-i}(V, V \setminus S) & \xrightarrow{T_{S,V,M}} & H^{2m-i}(M, M \setminus S) \\ \downarrow A_{V,S} & & \wr\downarrow A_{M,S} \\ H_i(S) & \xrightarrow{=} & H_i(S). \end{array} \tag{10.4.10}$$

In the sequel, we consider either the case S is a compact connected subcomplex in the regular part $V_{\mathrm{reg}} = V \setminus \mathrm{Sing}(V)$ or the case S is a compact connected component of $\mathrm{Sing}(V)$. In the former case, $A_{V,S}$ is the Alexander isomorphism discussed in Sect. 1.3. In the latter case, let $\widehat{U}$ be an open neighborhood of S in M, denote $U = \widehat{U} \cap V$, and assume that $U \setminus S$ is in V_{reg}. We let $\widehat{\mathcal{T}} \subset \widehat{U}$ be a tube, union of (D)-cells which are dual of (K)-simplices contained in S, as in 1.1.2. We can assume $\partial\widehat{\mathcal{T}}$ is transverse to V_{reg} and set $\mathcal{T} = \widehat{\mathcal{T}} \cap V$. We write $\partial\mathcal{T} = V \cap \partial\widehat{\mathcal{T}}$, which is a hypersurface in V_{reg}. With these, we apply the above homomorphisms to the pairs $(\widehat{U}, \widehat{U} \setminus S) \simeq (\widehat{\mathcal{T}}, \partial\widehat{\mathcal{T}})$ and $(U, U \setminus S) \simeq (\mathcal{T}, \partial\mathcal{T})$.

10.5 Localization of the Schwartz Classes

In this section we follow [31]. We denote by S a compact connected (K)-subcomplex of V such that $S \cap D^{(2p)}$ is either a subset of the regular part $V_0 = V_{\mathrm{reg}}$ or a component of $\mathrm{Sing}(V)$. We are writing $p = n - r + 1$ and $q = m - r + 1$, where $n = \dim_{\mathbb{C}} V$ and $m = \dim_{\mathbb{C}} M$, thus we have $q - p = m - n$. Let us denote by U a neighborhood of S in V such that $U \setminus S$ still intersects $(D)^{(2p)}$ in V_0.

10.5.1 The Topological Viewpoint

It follows from the above discussion that there exist stratified r-fields on $(D)^{(2q)} \cap U$ whose singularities are all located on S. Let $v_1^{(r)}$ and $v_2^{(r)}$ be two such r-fields and let us consider a tube $\mathcal{T}$ in U around S. There is a well defined secondary characteristic class $d(v_1^{(r)}, v_2^{(r)}) \in H^{2p-1}(\partial\mathcal{T})$ called the *difference* and defined as in Sect. 1.3.2. Let $\delta : H^{2p-1}(\partial\mathcal{T}) \longrightarrow H^{2p}(\mathcal{T}, \partial\mathcal{T})$ be the connecting homomorphism and let $A_V : H^{2p}(\mathcal{T}, \partial\mathcal{T}) \longrightarrow H_{r-1}(S)$ be the Alexander homomorphism. We set

$$d_S(v_1^{(r)}, v_2^{(r)}) = A_V \delta\, d(v_1^{(r)}, v_2^{(r)}).$$

Definition 10.5.1. For an r-frame $v^{(r)}$ on $(D)^{(2q)} \cap (U \setminus S)$, we define the *Schwartz class* $\mathrm{Sch}(v^{(r)}, S)$ of $v^{(r)}$ at S to be the class in $H_{2r-2}(S)$ given by:

$$\mathrm{Sch}(v^{(r)}, S) = \begin{cases} \mathrm{PH}(v^{(r)}, S) & \text{if } S \cap (D)^{(2q)} \subset V_0, \\ c_{r-1}(S) + d_S(v_{\mathrm{rad}}^{(r)}, v^{(r)}) & \text{if } S \subset \mathrm{Sing}(V), \end{cases}$$

where $v_{\mathrm{rad}}^{(r)}$ is a frame constructed by radial extension, PH is the Poincaré–Hopf class defined in Definition 1.3.4 for $S \subset V_0$, and $c_{r-1}(S)$ is the homological Schwartz class.

In particular for the frame $v_{\mathrm{rad}}^{(r)}$, $\mathrm{Sch}(v_{\mathrm{rad}}^{(r)}, S) = c_{r-1}(S) \in H_{2r-2}(S)$. We also remark that if $r = 1$, the class $\mathrm{Sch}(v_{\mathrm{rad}}^{(r)}, S)$ is the Schwartz index of the vector field defined in Chap. 2.

From the definition we get that for two r-frames $v_1^{(r)}$ and $v_2^{(r)}$ on $(D)^{(2p)} \cap (U \setminus S)$,

$$\mathrm{Sch}(v_2^{(r)}, S) = \mathrm{Sch}(v_1^{(r)}, S) + d_S(v_1^{(r)}, v_2^{(r)}). \tag{10.5.1}$$

Let us consider now a neighborhood U of $\mathrm{Sing}(V)$ in V. We know already that there exist stratified r-fields on $(D)^{(2q)} \cap U$ whose singularities are all in $\mathrm{Sing}(V)$. Elementary obstruction theory [153] then tells us that every such r-field can be extended to all of $(D)^{(2q)} \cap V_0$ with a singular set which is a subcomplex of V_0. More generally, let Σ be a compact (K)-subcomplex in V_0 disjoint from a neighborhood U_1 of $\mathrm{Sing}(V)$ in V. We denote by (S_λ) the connected components of $\mathrm{Sing}(V) \cup \Sigma$ and set $V^* = V \setminus U_1$. Let i_λ and ι be the inclusions $S_\lambda \hookrightarrow V$ and $V^* \hookrightarrow V$, respectively. The second one induces a homomorphism ι_* in homology with compact supports. The following theorem is similar to formula 1.1.3, the proof is easy using 10.5.1 above.

Theorem 10.5.2. *Let V be a compact complex analytic n-variety embedded in a complex m-manifold M and let Σ be a subcomplex in V_0 as above. For every stratified r-frame $v^{(r)}$ on $(D)^{(2q)} \cap (V_0 - \Sigma)$, $q = m - r + 1$, we have*

$$\sum_{\lambda}(i_\lambda)_*\mathrm{Sch}(v^{(r)}, S_\lambda) = c_{r-1}(V).$$

Thus, decomposing the previous summation according to the fact that S_λ *is in* $\mathrm{Sing}(V)$ *or in* Σ*, we get :*

$$\sum_{S_\lambda\subset \mathrm{Sing}(V)}(i_\lambda)_*\mathrm{Sch}(v^{(r)}, S_\lambda) + \iota_* c_{r-1}(V^*; v^{(r)}) = c_{r-1}(V),$$

where the sum is taken over the connected components of $\mathrm{Sing}(V)$*. In particular, for a radial* r*-frame* $v^{(r)}_{\mathrm{rad}}$*, we have:*

$$c_{r-1}(V) = \sum_{S_\lambda\subset \mathrm{Sing}(V)}(i_\lambda)_* c_{r-1}(S_\lambda) + \iota_* c_{r-1}(V^*; v^{(r)}_{\mathrm{rad}}).$$

Remark 10.5.1. In other words this theorem is telling us that to define the Schwartz class $c^q(V) \in H^{2q}(M, M \setminus V; \mathbb{Z})$ we may consider any stratified r-frame $v^{(r)}$, $q = m - r + 1$, on $(D)^{(2q)} \cap \widehat{U}$, then $c^q(V)$ is the obstruction to extending it to a stratified r-frame on $V \cap (D)^{(2q)}$. The contributions for $c^q(V)$ are splitted in two parts. On one hand we have the contribution of the regular part $\iota_* c_{r-1}(V^*; v^{(r)})$; this is the usual Chern class of V minus an open regular neighborhood of $\mathrm{Sing}(V)$ relative to the choice of frame $v^{(r)}$ on its boundary (in the appropriate skeleton). On the other hand we have the individual contributions of each connected component of the singular set $(i_\lambda)_*\mathrm{Sch}(v^{(r)}, S_\lambda)$. Each of these depends on the choice of frame, but their total sum is $c^q(V)$ independent of the frame, a result in the spirit of the Poincaré–Hopf Theorem for vector fields.

Remark 10.5.2. Again, just as in 2.4.4, one may consider a stratified r-frame $v^{(r)}$ on the intersection of $(D)^{(2q)}$ with a neighborhood $\widehat{U} \subset M$ of a connected component $S \subset \mathrm{Sing}(V)$ whose singularities are all in S. We have defined above a localization $\mathrm{Sch}(v^{(r)}, S)$ of the Schwartz class c_{r-1} at S. But in the previous section we defined a local index for $v^{(r)}$ at each isolated singularity. Taking into account only the singularities of $v^{(r)}$ in S we get another localization of c_{r-1} at S, say $\widehat{\mathrm{Sch}}(v^{(r)}, S)$. The proof of 2.4.4 can be easily adapted to this case, thus showing that both localizations of the Schwartz class coincide.

10.5.2 The Differential Geometric Viewpoint

We consider a (D)-cellular tube $\widehat{\mathcal{T}}$ around S in $\widehat{U}$ as before. Let us denote by (D') the cellular decomposition of M dual to a barycentric subdivision (K') of (K). The cells of (D') consist of simplices of the second barycentric subdivision (K'') of (K). We denote by $\widehat{\mathcal{R}}$ the (D')-cellular tube around S. Thus $\widehat{\mathcal{R}}$ is in the interior of $\widehat{\mathcal{T}}$ and the (D)-cells are transverse to $\partial\widehat{\mathcal{R}}$. We endow R_M with the ordinary orientation as the boundary. We set $\mathcal{R} = \widehat{\mathcal{R}} \cap V$.

Suppose we have an r-frame $v^{(r)}$ on $(U \setminus S) \cap D^{(2q)}$, $q = m - r + 1$, we may describe the Schwartz class $\mathrm{Sch}(v^{(r)}, S)$ of $v^{(r)}$ at S as follows.

First we consider the case where S is in the regular part V_0 of V (thus U is also in V_0) and give a differential geometric interpretation of the Poincaré–Hopf class $\mathrm{PH}(v^{(r)}, S)$. The relative class $c^p(\mathcal{T}, \partial\mathcal{T}; v^{(r)})$ defined as in 10.3.0 is now defined by taking an "$v^{(r)}$-trivial connection" for TU away from S. To be more precise, let ∇ be a connection for TU on U and let ∇_0 be an $v^{(r)}$-trivial connection for TU on a neighborhood of $(U \setminus S) \cap D^{(2q)}$ in U. Here ∇_0 being $v^{(r)}$-trivial means that $\nabla_0(v) = 0$ for every member v of $v^{(r)}$ so that $c^p(\nabla_0) = 0$ (see, for example, [156, Ch.II, 9],). Then the image ξ of $c^p(\mathcal{T}, \partial\mathcal{T}; v^{(r)})$ by the Thom–Gysin homomorphism $\tau : H^{2p}(U, U \setminus S) \to H^{2q}(\widehat{U}, \widehat{U} \setminus S)$ is represented by the cocycle

$$\gamma \mapsto \int_{\gamma \cap \mathcal{R}} c^p(\nabla) + \int_{\gamma \cap \partial\mathcal{R}} c^p(\nabla, \nabla_0), \tag{10.5.3}$$

for a relative cycle $\gamma \in C_{2q}^{(D)}(\widehat{\mathcal{T}}, \partial\widehat{\mathcal{T}})$ [108, 109], where $C_k^{(D)}(A)$ denotes the chains of dimension k in the (D)-complex A. The Poincaré–Hopf class $\mathrm{PH}(v^{(r)}, S)$ is then given by $A_V\, c^p(\mathcal{T}, \partial\mathcal{T}; v^{(r)}) = A_M\, \xi$ (see (10.4)).

Now suppose S may be a component of $\mathrm{Sing}(V)$ and let $v_{\mathrm{rad}}^{(r)}$ be a radial r-frame on $(\widehat{U} \setminus S) \cap D^{(2q)}$. Recall that the Schwartz class of $v_{\mathrm{rad}}^{(r)}$ at S is given by $\mathrm{Sch}(v_{\mathrm{rad}}^{(r)}, S) = c_{r-1}(S) = A_M\, c^q(S)$, where $c^q(S) \in H^{2q}(\widehat{\mathcal{T}}, \partial\widehat{\mathcal{T}}) \simeq H^{2q}(\widehat{U}, \widehat{U} \setminus S)$ is the q-th Schwartz class of S. We may assume that $v_{\mathrm{rad}}^{(r)}$ is given on a neighborhood $\widehat{W}$ of $(\widehat{U} \setminus S) \cap D^{(2q)}$. We denote by $\widehat{\nabla}$ a connection for TM on $\widehat{U}$ and by $\widehat{\nabla}_0$ an $v_0^{(r)}$-trivial connection for TM on $\widehat{W}$. From the definitions, we have the following.

Proposition 10.5.1. *The relative class $c^q(S)$ is represented by the cocycle*

$$\gamma \mapsto \int_{\gamma \cap \widehat{\mathcal{R}}} c^q(\widehat{\nabla}) + \int_{\gamma \cap \partial\widehat{\mathcal{R}}} c^q(\widehat{\nabla}, \widehat{\nabla}_0),$$

for a relative cycle $\gamma \in C_{2q}^{(D)}(\widehat{\mathcal{T}}, \partial\widehat{\mathcal{T}})$.

A differential geometric description for the Schwartz class of a general frame is obtained by combining the above and the following formula for the difference cocycle, introduced earlier (see Sect. 1.3.2). Let S be either a compact connected (K)-subcomplex in V_0 or a connected component of $\mathrm{Sing}(V)$ as before. Let $v_1^{(r)}$ and $v_2^{(r)}$ be two r-frames on $(U \setminus S) \cap D^{(2q)}$. We may assume that $v_1^{(r)}$ and $v_2^{(r)}$ are given on a neighborhood W of $(U \setminus S) \cap D^{(2q)}$

in U. For each $i = 1, 2$, let ∇_i be an $v_i^{(r)}$-trivial connection for TV_0 on W. We refer [31, Lemma 3.4] for the proof of the following

Lemma 10.5.1. *The difference* $\delta d(v_1^{(r)}, v_2^{(r)})$ *is in* $H^{2p}(U, U \setminus S)$ *whose image by the Thom–Gysin homomorphism* $\tau : H^{2p}(U, U \setminus S) \to H^{2q}(\widehat{U}, \widehat{U} \setminus S)$ *is represented by the cocycle*

$$\gamma \mapsto \int_{\gamma \cap \partial \mathcal{R}} c^p(\nabla_1, \nabla_2),$$

for a relative cycle $\gamma \in C_{2q}^{(D)}(\widehat{\mathcal{T}}, \partial \widehat{\mathcal{T}})$*.*

10.6 MacPherson and Mather Classes

Let us discuss briefly the MacPherson classes (see [117], [28]). We use the local Euler obstruction, defined in Chap. 8. This obstruction satisfies the following property: there exists (unique) integers $\{n_\alpha\}$ for which the equation

$$\sum n_\alpha \operatorname{Eu}_{\overline{V}_\alpha}(x) = 1 \tag{10.6.1}$$

is satisfied for all points x in V, where the sum runs over all strata V_α containing x in their closure. This statement is obvious for points in the regular stratum, for the points in $\operatorname{Sing}(V)$ (10.6.1) can be easily proved by induction.

Consider now (see Chap. 8) the Nash blow up $\widetilde{V} \xrightarrow{\nu} V$ of V, the Nash bundle $\widetilde{T} \xrightarrow{\pi} \widetilde{V}$, and the Chern classes of $\widetilde{T}$, $c^j(\widetilde{T}) \in H^{2j}(\widetilde{V})$. The Poincaré homomorphism (in general not an isomorphism):

$$\beta_{\widetilde{V}} : H^{2j}(\widetilde{V}) \xrightarrow{\cap [\widetilde{V}]} H_{2n-2j}(\widetilde{V}),$$

carries these into homology classes which can be pushed forward into the homology of V via the homomorphism ν_* induced by the projection.

Definition 10.6.1. The *Mather classes* of V are:

$$c^{\mathrm{Ma}}_{n-j}(V) \;=\; \nu_*(c^j(\widetilde{T}) \cap [\widetilde{V}]) \;\in\; H_{2(n-j)}(\widetilde{V}) \quad , \; j = 0, ..., n.$$

This definition can be applied to every compact complex analytic space, in particular to the closure $\overline{V}_\alpha$ of each stratum $V_\alpha \subset V$, which is again complex analytic, and compact. One can define the Mather classes of $\overline{V}_\alpha$ in the same way as above. Since these classes live in homology, the inclusion $\overline{V}_\alpha \overset{\iota}{\hookrightarrow} V$ carries them into the homology of V.

Definition 10.6.2. The *MacPherson class* of degree $r-1$ is defined by:

$$c_{r-1}(V) = c^{\mathrm{Ma}}_{r-1}(\sum n_\alpha \overline{V}_\alpha) = \sum n_\alpha \, \iota_* c^{\mathrm{Ma}}_{r-1}(\overline{V}_\alpha),$$

where the n_α are the integers characterized by the equation 10.6.1.

The relation between Mather classes on one side and MacPherson classes on the other side follows form MacPherson's definition itself: his construction uses Mather classes, taking into account the values of the local Euler obstruction along the strata. The precise relation between Mather classes and Schwartz classes was determined in [33] (see 10.6.2 below), and this is a key point for the identification of Schwartz and MacPherson classes. The MacPherson classes live in $H_*(V)$. It is proved in [33] that the Alexander homomorphism $H^j(M, M \setminus V) \to H_{2m-j}(V)$ carries the Schwartz classes into MacPherson's. A key step in that proof is the Proportionality Theorem of [33], that we discussed in Sect. 10.2.

Notice that the Schwartz indices of frames determine the elementary cocycle in 10.3 that defines the corresponding Schwartz class. Thus the theorem above establishes a deep connection between Schwartz classes and Mather classes. This relation is made precise in the following theorem of [33, Th. 4.1] that we state without proof (for a complete proof, see [28]). Recall that the Euler obstruction is constant on Whitney strata ([33, 10.2]), so we write Eu_{V_α} to denote the Euler obstruction at points in the stratum V_α.

Theorem 10.6.2. *Let $v^{(r)}$ be a frame as in 10.3, the Schwartz–MacPherson class $c_{r-1}(V) \in H_{2(r-1)}(V)$ is represented by the cycle*

$$\sum_{s_i \subset V, \dim s_i = 2(r-1)} \mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}, a_i) \cdot s_i,$$

where a_i denotes the barycenter of s_i. The Mather class $c^{\mathrm{Ma}}_{r-1}(V)$ of degree $(r-1)$ is represented in $H_{2(r-1)}(V)$ by the cycle

$$\sum_{s_i \subset V, \dim s_i = 2(r-1)} \mathrm{Eu}_V(a_i) \cdot \mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}, a_i) \cdot s_i$$

Once we have this expression for a cycle representing the Mather classes as weighted Schwartz classes, the identification of Schwartz and MacPherson classes follows (see [28, 33]).

Since the definition of Schwartz classes and MacPherson classes is completely different, the fact that they coincide brings a deep richness into the subject. Schwartz classes are geometrically defined and allow us to understand what these classes measure in terms of obstruction theory, but they are hard to work with. On the contrary MacPherson classes are in a way more difficult to grasp, but they have useful factorial properties. This makes them be powerful invariants and easy to work with. These important invariants of singular varieties have been widely studied by many authors. We refer to [28] for a complete account on the subject.

In the sequel we refer to these as Schwartz–MacPherson classes, or just SM-classes for short, and we denote them c_*^{SM}. The class c_i^{SM} lives in $H_{2i}(V;\mathbb{Z})$, $i = 0, \cdots, n$.

Chapter 11
The Virtual Classes

Abstract The constructions described in the previous chapter, mostly based on [31, 33, 139], provide geometric insights of the Schwartz–MacPherson classes via obstruction theory and localization. These approaches are useful for understanding what the classes measure from the viewpoint of indices of vector fields and frames. The Fulton–Johnson classes [59, 60] provide another way of generalizing the Chern class of complex manifolds to the case of singular varieties. In the context we consider, they coincide with the virtual classes (see Sect. 11.1).

In this chapter we define and study the virtual classes from a viewpoint similar to the one we used in the previous chapter for the Schwartz–MacPherson classes. This is based on our articles [31, 34], joint work with D. Lehmann.

If the variety V is globally defined by a function on M, the virtual classes can be localized topologically and one can interpret them as "weighted" Schwartz classes. That is explained in Sect. 11.3 where we prove the Proportionality theorem of [34] for this index. This theorem is analogous to, and inspired by, the similar theorem of [33] for the Schwartz index, proved in the previous chapter.

In Sect. 11.4, the localization of virtual classes is performed using the differential geometric method of [31], *i.e.*, Chern–Weil theory, and using stratified frames. In that context, we construct localized "Fulton–Johnson classes" at the singular set of the given frames. While Sects. 11.1–11.3 are of a local nature, Sect. 11.4 is global.

11.1 Virtual Classes

We recall from Chap. 5 that if V is a compact local complete intersection of dimension n in a manifold M of dimension $m = n + k$, defined as the zero set of a holomorphic section s of a holomorphic vector bundle N of rank k over M, then V has the virtual tangent bundle $\tau_V = [TM - N]|_V$ regarded as an element in complex K-theory $KU(V)$.

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The total Chern class of the virtual tangent bundle is defined in the usual way:

$$c^*(\tau_V) = c^*(TM|_V) \smile c^*(N|_V)^{-1} \quad \text{in} \quad H^*(V)\,.$$

The p^{th} Chern class of τ_V is the component of $c^*(\tau_V)$ in dimension $2p$, for $p = 1, ..., n$. That is, $c^p(\tau_V)$ is the coefficient of t^p in the expansion of

$$(1 + \sum_{i=1}^{m} t^i c^i(TM|_V))\,(1 + \sum_{j=1}^{k} t^j c^j(N|_V))^{-1}.$$

Definition 11.1.1. *The virtual cohomology (Chern) classes* of the local complete intersection V, denoted $c^p_{\text{vir}}(V)$, are the Chern classes of the virtual tangent bundle τ_V. One has also the corresponding *total virtual class* $c^*_{\text{vir}}(V) = 1 + c^1_{\text{vir}}(V) + \cdots + c^n_{\text{vir}}(V)$. The *homology virtual class* of V of degree p is the image of the virtual class $c^{n-p}_{\text{vir}}(V)$ under the Poincaré homomorphism $H^{2n-*}(V) \to H_*(V)$.

Theorem 11.1.1. *The total homology virtual class of V coincides with the Fulton–Johnson "canonical" class of V defined in [59, 60], we denote it by $c^{\text{FJ}}_*(V) \in H_{2*}(V)$.*

In the sequel we often refer to the components of $c^{\text{FJ}}_*(V)$ as *FJ-classes* and denote them $c^{\text{FJ}}_i(V) \in H_{2i}(V;\mathbb{Z})$. The book of Fulton [59], as well as several articles of Aluffi, and Parusiński–Pragacz explore these classes from the algebraic geometry viewpoint. Here we are mostly concerned with the topological and differential-geometric aspects of the theory, particularly in their relations with indices of vector field and frames, and that is the subject we explore in that chapter (see also [28] for the algebraic-topology viewpoint).

There are some special cases in which the geometry and topology of these classes is particularly apparent. To explain this, consider first the case where the variety V has isolated singularities. The 0-degree FJ-class is the image under the Alexander homomorphism of the top Chern class $c^n(\tau_V)$ of the virtual tangent bundle; we know already from Chap. 5 that this class, evaluated on the orientation cycle $[V]$, equals the total virtual index of every vector field on V with isolated singularities. Since for isolated singularities the virtual index coincides with the GSV index, this means that this FJ-class equals the Euler–Poincaré characteristic of a smoothing of V, *i.e.*, the (almost complex) manifold $V^\#$ obtained by cutting off neighborhoods of the singular points and replacing them by local Milnor fibers. Given a vector field v on $V^\#$ with isolated singularities (whose zero set represents a 0-dimensional homology class which is the Poincaré-dual of its top Chern class $c^n(V^\#)$), one has that as $V^\#$ degenerates to V, the vector field v degenerates to a vector field on V and its zeros determine the 0-degree FJ-class.

It turns out that under certain conditions, which are stringent but still rather general, this is what happens for all FJ-classes: if the variety V can be embedded in an analytic family of complex manifolds V_t that degenerate to V, then the FJ-classes of V "correspond" to the homology Chern classes of the V_t. This was first observed in [129].

Let us explain this more carefully. Assume V is defined by a regular holomorphic section s of a vector bundle E over a complex manifold M as before. The singular set $\mathrm{Sing}(V)$ is the set of points where s is not transverse to the zero-section. Assume further that there exists a family $\{s_t\}$, $t \in \mathbb{C}$, such that for $t = 0$ the section s_0 is s, and for $t \neq 0$ the section s_t is transverse to the zero-section, so its zero set is a complex submanifold V_t of M. Each V_t has its usual Chern classes $c^j(V_t) \in H^{2j}(V_t)$, and Poincaré duality on V_t carries them into homology classes $c_i(V_t) \in H_{2i}(V_t)$, $i = n - j$. These are usually called *the homology Chern classes* of the manifold in question. By compactness, given a tube $\mathcal{T}$ around V in M, there is an $\varepsilon > 0$ sufficiently small so that if $0 < |t| < \varepsilon$ then the manifold V_t is fully contained in $\mathcal{T}$. The inclusion $V_t \xrightarrow{\iota} \mathcal{T}$ induces a morphism in homology $H_*(V_t) \xrightarrow{\iota_*} H_*(\mathcal{T})$. The tube $\mathcal{T}$ has V as a deformation retract, so one has also a morphism $H_*(\mathcal{T}) \xrightarrow{r_*} H_*(V)$. One has:

Theorem 11.1.2. [129] *The morphism*

$$r_* \circ \iota_* \, : H_*(V_t) \longrightarrow H_*(V),$$

carries the homology Chern classes of V_t into the FJ-classes of V.

The proof of this result is in fact an immediate consequence of the invariance of Chern classes under specialization.

11.2 Lifting a Frame to the Milnor Fiber

In this section we define an index for stratified frames generalizing the GSV index defined in Chap. 3 for vector fields on hypersurface germs with nonisolated singularities.

Let us denote by U an open neighborhood of the origin in $\mathbb{C}^{n+1}$, $n > 0$ and let us consider a holomorphic function

$$f : (U, 0) \longrightarrow (\mathbb{C}, 0),$$

defining a hypersurface germ $V := f^{-1}(0)$. We know from [128] that there is a Whitney stratification $\{V_\alpha\}_{\alpha \in A}$ of V that satisfies the strict Thom *w_f-condition* for all strata (see 3.5.3).

Then one has a locally trivial fiber bundle (see Chap. 3):

$$f : \Big(\big(\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\delta)\big) \setminus f^{-1}(0) \Big) \longrightarrow (\mathbb{D}_\delta \setminus \{0\}) \subset \mathbb{C},$$

for every $\varepsilon > 0$ sufficiently small and $\delta = \delta(\varepsilon) > 0$ sufficiently small with respect to ε, where $\mathbb{B}_\varepsilon$ is a small ball around $0 \in \mathbb{C}^{n+1}$ and $\mathbb{D}_\delta$ is a small ball around $0 \in \mathbb{C}$.

We know already from Chap. 3 that given a stratified vector field v on V with an isolated singularity at a point $x \in V$, we can lift v to a local Milnor fiber $\mathbf{F}(x)$ of V at x and the Poincaré–Hopf index of this lifting to $\mathbf{F}(x)$ is well defined and gives an invariant of v at $x \in V$ that we called the GSV index. We proved that if v is obtained by radial extension then its GSV index at x is proportional to the Schwartz index of v at x, the proportionality factor being the Euler characteristic of $\mathbf{F}(x)$ (Theorem 3.6.1). We are going to make the analogous constructions for frames.

Let us recall the *tube map* $\pi : \mathbf{F} \to V$, from a local Milnor fiber into V, introduced in 3.5.3, that we use to lift vectors from V to $\mathbf{F}$. We do it only for hypersurfaces. Consider a small ball $\mathbb{B}_\varepsilon$ around $0 \in U$. Let ρ be a radial vector field in a sufficiently small (with respect to ε) disk $\mathbb{D}_\delta$ around $0 \in \mathbb{C}$, whose solutions are arcs converging to 0. We can assume further that for each $t \in \mathbb{D}_\delta \setminus \{0\}$ the (Milnor) fiber $\mathbf{F}_t = f^{-1}(t)$ intersects the boundary sphere $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$ transversely. Set $\mathcal{T} = f^{-1}(\mathbb{D}_\delta \setminus \{0\})$. As we know,

$$f|_{\mathcal{T}} \colon \mathcal{T} \longrightarrow \mathbb{D}_\delta \setminus \{0\}$$

is a locally trivial fiber bundle, and by [167] we can lift ρ to a rugose (hence integrable) vector field $\widehat{\rho}$ in $\mathcal{T}$, whose solutions are arcs that start in $\partial\mathcal{T} = f^{-1}((\mathbb{S}_\delta) \setminus \{0\})$, $\mathbb{S}_\delta = \partial\mathbb{D}_\delta$, they finish in V and they are transverse to all the "tubes" $f^{-1}(\mathbb{S}_\eta)$ with $\eta \in]0, \delta[$.

This vector field $\widehat{\rho}$ defines a C^∞ retraction ξ of $\mathcal{T}$ into V, with V as fixed point set. The restriction of ξ to any fixed Milnor fiber $\mathbf{F} = f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$, $t_0 \in \mathbb{S}_\delta$, provides a continuous map $\pi : \mathbf{F} \to V$ which is surjective and it is C^∞ over the regular part of V. As before, we call such map ξ, or also π, a *tube map* for V.

We use π to lift the stratified frame $v^{(r)}$ on V to an r-frame $\tilde{v}^{(r)}$ on $\mathbf{F}$. Given a point $x \in \mathbf{F}$, we let γ_x be the solution of $\widehat{\rho}$ that starts at x. The end-point of γ_x is the point $\pi(x) \in V$. We parameterize this arc γ_x by the interval $[0, 1]$, with $\gamma_x(0) = x$ and $\gamma_x(1) = \pi(x)$. We assume that this interval $[0, 1]$ is the arc in $\mathbb{D}_\delta$ going from t_o to 0, so that for each $t \in [0, 1[$, the point $\gamma_x(t)$ is in a unique Milnor fiber $\mathbf{F}_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon$. The family of tangent spaces to $\mathbf{F}_t$ at the points $\gamma_x(t)$ define a 1-parameter family of n-dimensional subspaces of $\mathbb{C}^{n+1}$, that converges to an n-plane $\Lambda_{\pi(x)} \subset T_{\pi(x)}(U)$ when t goes to 1; one has an induced isomorphism $T_x\mathbf{F}(x) \simeq \Lambda_{\pi(x)}$.

Since the stratification satisfies Thom's a_f-condition, $\Lambda_{\pi(x)}$ contains the space $T_{\pi(x)}V_\alpha$, tangent to the stratum that contains $\pi(x)$. Hence, the given frame $v^{(r)}$ can be lifted to a frame $\tilde{v}^{(r)}$ in $T_x\mathbf{F}(x)$. Thus, if the frame $v^{(r)}$ is defined without singularity on a set $A \subset V$, we obtain an r-frame $\tilde{v}^{(r)}$ over the inverse image $\pi^{-1}(A) \subset \mathbf{F}$. The fact that the stratification further satisfies the strict Thom's w_f-condition guarantees that this r-frame $\tilde{v}^{(r)}$ is continuous.

We summarize this discussion in the following proposition:

Proposition 11.2.1. *Let $v^{(r)}$ be a continuous r-frame defined without singularity on a subset $A \subset V$. Assume the map f that defines the complete intersection V satisfies the strict Thom w_f-condition at all points in V. Let $\mathbf{F}$ be a local Milnor fiber for V at 0 and let $\pi : \mathbf{F} \to V$ be a tube map. Then we can lift $v^{(r)}$ to a continuous r-frame $\tilde{v}^{(r)}$ on $\pi^{-1}(A) \subset \mathbf{F}$ using the map π.*

As before, we denote by $\{V_\alpha\}$ the strata of a stratification of $V \cap \mathbb{B}_\varepsilon$, restriction of a Whitney stratification of U to V, and we denote by $\{W_\beta\}$ a Whitney stratification of $\mathbf{F}$ such that:

(1) $\pi : \mathbf{F} \longrightarrow X \cap \mathbb{B}_\varepsilon$ is a stratified map,
(2) for every β, the restriction of π to W_β is a map of constant rank from W_β to a stratum V_α of X, where π is the tube map.

Such stratifications exist by [73]. We notice that each $\pi^{-1}(V_\alpha)$ is union of strata $\{W_\beta\}$.

In the case of isolated singularities, the construction by Lê [105] of "polyèdres d'effondrement" allows us to prove that there are triangulations of U and $\mathbf{F}$ compatible with the previous stratifications, and such that π is a simplicial map. For non necessarily isolated singularities, let us consider a triangulation (K) of X compatible with the stratification $\{V_\alpha\}$; as the restriction of π to each stratum $\{W_\beta\}$ of $\mathbf{F}$ has constant rank, the intersection of the inverse image of a simplex of (K) with the strata W_β can be decomposed into cells satisfying the following proposition:

Proposition 11.2.2. *There is a simplicial triangulation (K) of U compatible with the stratification $\{V_\alpha\}$ and a cellular decomposition $(\widetilde{K})$ of $\mathbf{F}$ compatible with the stratification $\{W_\beta\}$, such that for each cell $\widetilde{s}_\beta$ of $(\widetilde{K})$, there is a simplex s_α of (K) such that $\pi(\widetilde{s}_\beta) = s_\alpha$ and the restriction of π to each open cell $\widetilde{s}_\beta$ has constant rank.*

11.3 The Fulton–Johnson Classes

In Chap. 3 we stated and proved the Proportionality Theorem for the GSV index of a vector field. Here we extend that result to r-frames, $r \geq 1$. The method above for lifting a vector field from V to a local Milnor fiber works for frames. Using the notations introduced in 10.1, we have:

Theorem 11.3.1. (Proportionality Theorem for frames). *Let $v^{(r)}$ be an r-frame constructed by radial extension, with isolated singularities on the $2q$-cells σ_i, with index $I(v^{(r)}, a_i)$ at the barycenter $\{a_i\}$ of σ_i. Then the* *obstruction to the extension of* $\widetilde{v}^{(r)}$ *as a section of* $W_r(T\mathbf{F})$ *(see 1.3.1) on* $\widetilde{\beta}_i = \pi^{-1}(\sigma_i \cap V)$ *is*

$$\mathrm{Obs}\,(\widetilde{v}^{(r)}, W_r(T\mathbf{F}), \widetilde{\beta}_i) = \chi(\mathbf{F}_{a_i}) \cdot I(v^{(r)}, a_i).$$

The proof is similar to the one of Theorem 10.2.1 using the liftings defined in 3.6.1.

A corollary is the following expression of virtual classes:

Theorem 11.3.2. *Let us assume that* $V \subset M$ *is a hypersurface, defined by* $V = f^{-1}(0)$, *where* $f : M \to \mathbb{D}$ *is a holomorphic function into an open disk* $\mathbb{D}$ *around* 0 *in* $\mathbb{C}$. *For each point* $a \in V$, *let* $\mathbf{F}_a$ *denote a local Milnor fiber, and let* $\chi(\mathbf{F}_a)$ *be its Euler–Poincaré characteristic. Then the Fulton–Johnson class* $c^{\mathrm{FJ}}_{r-1}(V)$ *of* V *of degree* $(r-1)$ *is represented in* $H_{2(r-1)}(V)$ *by the cycle*

$$\sum_{\sigma_i \subset V,\, \dim \sigma_i = 2(r-1)} \chi(\mathbf{F}_{a_i})\; I(v^{(r)}, a_i)\, \cdot\, \sigma_i \tag{11.3.3}$$

For this, let us denote by V_t the fibers $f^{-1}(t)$, $t \neq 0$. This is a 1-parameter family of n-dimensional complex submanifolds of M that degenerate to V when $t = 0$.

Since for $t \neq 0$ each V_t is a smooth complex manifold, its Chern classes $c^i(V_t) \in H^{2i}(V_t)$ are well defined, and since it is compact, by Poincaré duality one can think of these as homology classes in $H_{2n-2i}(V_t)$, denoted by $c_{n-i}(V_t)$. The class in degree 0, corresponding to $c^n(V_t)$, is the Euler–Poincaré characteristic of V_t.

We notice that, by the compactness of V, given a regular neighborhood $\mathcal{N}$ of V in M, we can find t sufficiently small so that $V_t \subset \mathcal{N}$. Thus, one has a homomorphism,

$$i_* : H_*(V_t) \longrightarrow H_*(\mathcal{N}),$$

induced by the inclusion. One also has:

$$r_* : H_*(\mathcal{N}) \longrightarrow H_*(V),$$

induced by a retraction r from $\mathcal{N}$ into V. The composition:

$$\psi = r_* \circ i_* : H_*(V_t) \longrightarrow H_*(V)$$

is the *Verdier specialization map*. Notice that by construction, for each $x \in V$, ψ is induced by the degenerating map π of Sect. 2 above, which is now globally defined on all of V_t. In other words, the Verdier specialization map is in this case the homomorphism in homology induced by the map $\pi : V_t \to V$ defined (locally) in Sect. 2 above.

For each V_t, $t \neq 0$, one has that $[TV_t] = [TM - N]|_{V_t}$ in K-theory. Thus the Chern classes of V_t are those of the virtual bundle $[TM - N]|_{V_t}$. By [168], the homology specialization map ψ carries the Chern classes of $TM|_{V_t}$ and $N|_{V_t}$ into the Chern classes of $TM|_V$ and $N|_V$, respectively. Thus, as noticed in [131], one has:

$$c_*^{\mathrm{FJ}}(V) = \psi\, c_*(V_t). \tag{11.3.4}$$

Let $\widetilde{v}^{(r)}$ be, as before, a lifting to V_t via the degenerating map π, of a frame $v^{(r)}$ on the $2p$-skeleton of V with isolated singularities. The Chern class $c^p(V_t)$ is represented by the obstruction cocycle $\widetilde{\gamma}$ satisfying

$$\langle \widetilde{\gamma}, \widetilde{\beta}_i \rangle = \mathrm{Obs}\,(\widetilde{v}^{(r)}, W_r(T\mathbf{F})V_t, \widetilde{\beta}_i).$$

where $\widetilde{\beta}_i = \pi^{-1}(\sigma_i \cap V)$.

By Theorem 11.3.1 one has: $\langle \widetilde{\gamma}, \widetilde{\beta}_\alpha \rangle = \chi(\mathbf{F}_{a_{\sigma_\alpha}}) \cdot I(v^{(r)}, a_{\sigma_\alpha})$. We conclude, using the following observation and Proposition 11.2.2 (see [28]):

Lemma 11.3.1. *Let $\widetilde{\gamma}$ be a cocycle in $Z^{2p}(V_t)$ representing the Chern class $c^p(V_t)$ and let us denote $k_\alpha = \langle \widetilde{\gamma}, \widetilde{\beta}_\alpha \rangle$. Then the cycle $\sum k_\alpha \sigma_\alpha$ for $\sigma_\alpha \subset V$ and $\dim \sigma_\alpha = 2r-2$ is homologous to $\pi_*(\widetilde{\gamma} \cap [V_t])$ in $Z_{2(r-1)}(V)$and represents the Fulton–Johnson class $c_{r-1}^{\mathrm{FJ}}(V)$.*

11.4 Localization of the Virtual Classes

In order to define the Milnor classes in a later section, and to calculate them in some cases, we develop the localization theory of virtual classes in the framework of Chern–Weil theory adapted to stratifications and triangulations. Here we use the notation introduced in the previous sections.

We now suppose that V is a compact local complete intersection of dimension n in a complex manifold M of dimension m, defined as the zero set of a holomorphic section s of a holomorphic vector bundle N of rank $k = m - n$ over M. The restriction $N|_{V_0}$ coincides with the normal bundle N_{V_0} of $V_0 = V \setminus \mathrm{Sing}(V)$ in M and we have an exact sequence of vector bundles,

$$0 \longrightarrow TV_0 \longrightarrow TM|_{V_0} \xrightarrow{\pi} N_{V_0} \longrightarrow 0. \tag{11.4.1}$$

The virtual class of an r-frame is defined by localizing $c^p(\tau_V)$ by the frame, where τ_V denotes the virtual tangent bundle (Sect. 5.1). To be more precise, let us consider a subset S and suitable neighborhoods as in the previous sections. Let $v^{(r)}$ be an r-frame on $(U \setminus S) \cap D^{(2q)}$. Let ∇ and ∇' be connections for TM and N, respectively, on $\widehat{U}$ and set $\nabla^\bullet = (\nabla, \nabla')$. Also, let ∇_0 and ∇_0' be connections for TM and N, respectively, on a neighborhood W of $(U \setminus S) \cap D^{(2q)}$ in U such that the pair $\nabla_0^\bullet = (\nabla_0, \nabla_0')$ is compatible and ∇_0 is $v^{(r)}$-trivial. If we consider the $2q$-cochain given by

$$\gamma \mapsto \int_{\gamma\cap\mathcal{R}} c^p(\nabla^\bullet) + \int_{\gamma\cap\partial\mathcal{R}} c^p(\nabla^\bullet, \nabla_0^\bullet), \qquad \gamma \in C_{2q}^{(D)}(\widehat{\mathcal{T}}, \partial\widehat{\mathcal{T}}), \quad (11.4.2)$$

it is a cocycle independent of the choices of connections, where ∇_0 is $v^{(r)}$-trivial, and defines an element η in $H^{2q}(\widehat{U}, \widehat{U} \setminus S)$.

Definition 11.4.1. We define the *virtual class* $\mathrm{Vir}(v^{(r)}, S)$ of $v^{(r)}$ at S to be the image of η by the Alexander isomorphism $A_M : H^{2q}(\widehat{U}, \widehat{U} \setminus S) \to H_{2r-2}(S)$.

Recall that, if S is in V_0, the Poincaré–Hopf class $\mathrm{PH}(v^{(r)}, S) \in H_{2r-2}(S)$ is dual to the class represented by the cocycle (10.5.3). Thus in this case, from Proposition 5.2.1 and a similar statement for the difference forms, we have

$$\mathrm{Vir}(v^{(r)}, S) = \mathrm{PH}(v^{(r)}, S).$$

The following formula for two r-frames $v_1^{(r)}$ and $v_2^{(r)}$ as above, analogous to (10.5.1), is a consequence of Lemma 10.5.1 and Proposition 5.2.1.

$$\mathrm{Vir}(v_2^{(r)}, S) = \mathrm{Vir}(v_1^{(r)}, S) + d_S(v_1^{(r)}, v_2^{(r)}). \quad (11.4.3)$$

Also the following theorem, analogous to Theorem 10.5.2, follows from the previous discussion.

Theorem 11.4.4. *Let Σ be a subset in V_0 as in Theorem 10.5.2. With the above hypotheses and notation, if $v^{(r)}$ is an r-frame on $(V_0 \setminus \Sigma) \cap D^{(2p)}$, we have, in $H_{2r-2}(V)$,*

$$\sum_{S\subset \mathrm{Sing}(V)} i_* \mathrm{Vir}(v^{(r)}, S) + \iota_* c_{r-1}(V^*, v^{(r)}) = c_{r-1}^{\mathrm{FJ}}(V),$$

where the sum is taken over the connected components of the singular set $\mathrm{Sing}(V)$ and $c_{r-1}(V^*; v^{(r)})$ is the Chern class of V^* relative to $v^{(r)}$, so that

$$\iota_* c_{r-1}(V^*; v^{(r)}) = \sum_{S\subset\Sigma} i_* \mathrm{PH}(v^{(r)}, S).$$

Chapter 12
Milnor Number and Milnor Classes

Abstract Both Schwartz–MacPherson and Fulton–Johnson classes generalize Chern classes to the case of singular varieties. It is known that for local complete intersections with isolated singularities, the 0-degree SM and FJ classes differ by the local Milnor numbers [149] and all other classes coincide [155]. As we explain in the sequel, if V has nonisolated singularities, the difference $c_i^{SM}(V) - c_i^{FJ}(V)$ of the SM and FJ classes is, for each i, a homology class with support in the homology $H_{2i}(\mathrm{Sing}(V))$ of the singular set of V. That is the reason for which their difference was called in [30, 31] the *Milnor class* of degree i. These classes have been also considered, from different viewpoints, by other authors, most notably by P. Aluffi, T. Ohmoto, A. Parusiński, P. Pragacz, J. Schürmann, S. Yokura.

In this chapter we introduce the Milnor classes of a local complete intersection V of dimension $n \geq 1$ in a complex manifold M, defined by a regular section s of a holomorphic bundle N over M. The aim of this chapter is to show that, as mentioned above, the Milnor classes are localized at the connected components of the singular set of V: If S is such a component then one has Milnor classes $\mu_i(V, S)$ of V at S in degrees $i = 0, \cdots, \dim S$. The 0-degree class coincides with the *generalized Milnor number* of V at S, introduced by Parusiński in [127] (if V is a hypersurface in M). The sum of all the Milnor classes over the connected components of $\mathrm{Sing}(V)$ gives the global Milnor classes studied in [8, 126, 131, 169]. See [28] for another presentation.

The method we use for constructing the localized Milnor classes comes from [31] and uses Chern–Weil theory. The idea is to use stratified frames to localize at the singular set the Schwartz–MacPherson and the Fulton–Johnson classes, in such a way that the difference of these localizations is canonical.

J.-P. Brasselet et al., *Vector Fields on Singular Varieties*,
Lecture Notes in Mathematics 1987, DOI 10.1007/978-3-642-05205-7_12,

12.1 Milnor Classes

For most authors, Milnor classes are globally defined as elements in $H_*(V, \mathbb{Z})$, on the other hand in [31], these classes are localized at the singular set of V from the beginning. We explain this in a moment, first we introduce the global classes; there is one such class in each degree:

Definition 12.1.1. For each $r = 0, 1, \cdots, n-1$, the r-th *Milnor class* $\mu_r(V)$ of V is:

$$\mu_r(V) \;=\; (-1)^{n+1}\big(c_r^{\mathrm{SM}}(V) \;-\; c_r^{\mathrm{FJ}}(V)\big) \quad \text{in} \quad H_{2r}(V, \mathbb{Z}).$$

The difference class

$$\mu_*(V) = (-1)^{n-1}(c_*^{\mathrm{SM}}(V) - c_*^{\mathrm{FJ}}(V))$$

is called the (total) Milnor class of V.

In fact, FJ-classes and SM-classes coincide with the usual Chern classes in the regular part of V. Thus Milnor classes ought to be concentrated in the singular set $\mathrm{Sing}(V)$. The results of [31,129] prove that this is indeed the case. Since the results of [149, 155] prove that in the case of isolated singularities this contribution corresponds to the local Milnor number at each singular point, and this is a local invariant of the singularity (not a global one), we considered in [31] *Milnor classes localized* at the connected components of the singular set of V. For each connected component S of $\mathrm{Sing}(V)$, the r-th *Milnor class* $\mu_r(V, S)$ of V at S is a homology class in $H_{2r}(S, \mathbb{Z})$. There is one such class for each $r = 0, 1, \cdots, s$, where s is the dimension of the component S. The inclusion $S \hookrightarrow V$ maps the homology of S into that of V, and adding up the contributions in each dimension of all the connected components of $\mathrm{Sing}(V)$ we get the corresponding global Milnor classes.

For hypersurfaces, the 0-degree localized Milnor class $\mu_0(V, S) \in H_0(S) \simeq \mathbb{Z}$ coincides with the *generalized Milnor number* of Parusiński [127], that we will discuss in Sect. 12.4. Thus $\mu_0(V, S)$ can be also considered as a generalized Milnor number for complete intersections.

Each connected component S has a contribution $\mu_r(V, S)$ to the global Milnor class $\mu_r(V)$ up to the dimension of S. Therefore, if $\mathrm{Sing}(V)$ has dimension 0, then all Milnor classes vanish in dimensions $r > 0$, *i.e.*, the SM and FJ classes coincide for all $r > 0$. If some component has dimension 1, then we have corresponding Milnor classes in dimensions 0, 1, and so on.

Since for isolated singularities the "Milnor classes" are just the Milnor numbers, which can be regarded as the number of vanishing cycles in the local Milnor fibers, it was natural to ask in [31] whether Milnor classes are related to the *vanishing homology*. Answers were given in [31] in particular cases, one of them is the Lefschetz type Theorem 12.3.1.

12.2 Localization of Milnor Classes

Let V be a local complete intersection of dimension n defined by a section of a vector bundle N over the ambient complex manifold M of dimension $m = n + k$, as in the previous section. We introduce the Milnor classes of V at a connected component S of $\mathrm{Sing}(V)$. For $r \geq 1$, let $v^{(r)}$ be an r-frame on $(U \setminus S) \cap D^{(2q)}$, where U is a neighborhood of S in V such that $U \setminus S \subset V_0$ and $q = m - r + 1$.

Definition 12.2.1. The $(r-1)$-st *Milnor class* $\mu_{r-1}(V, S)$ of V at S is defined by

$$\mu_{r-1}(V,S) = (-1)^{n+1}\Big(\mathrm{Sch}(v^{(r)}, S) - \mathrm{Vir}(v^{(r)}, S)\Big) \qquad \text{in} \quad H_{2r-2}(S),$$

which is independent of the choice of $v^{(r)}$ by (10.5.1) and (11.4.3).

We call $\mu_*(V,S) = \sum_{r\geq 0} \mu_r(V,S) \in H_*(S)$ the total Milnor class of V at S. Note that $\mu_r(V,S) = 0$ for $r > \dim_{\mathbb{C}} S$. Since there exist always frames as in Theorems 10.5.2 and 11.4.4, we have:

Theorem 12.2.1. *For a subvariety V of a complex manifold M as above,*

$$c_*(V) = c_*^{\mathrm{FJ}}(V) + (-1)^{n+1} \sum_S i_*\mu_*(V,S) \qquad \textit{in} \quad H_*(V),$$

where the sum is taken over the connected components S of $\mathrm{Sing}(V)$.

In particular, if the singularities of V are isolated points, then the Milnor classes are zero, except in degree 0 where they coincide with the usual Milnor numbers of [79,116,121]. Hence, in this case the SM classes and the FJ classes of V coincide in all dimensions, except in degree 0, where their difference is given by the sum of the usual Milnor numbers, recovering the formula in [149, 155].

Remark 12.2.1. 1. The classes $\mathrm{PH}(v^{(r)}, S)$, $\mathrm{Sch}(v^{(r)}, S)$ and $\mathrm{Vir}(v^{(r)}, S)$ may be defined for an r-frame $v^{(r)}$ on the intersection of a neighborhood of $\partial\mathcal{T}$ (in V) and $D^{(2q)}$, where $\mathcal{T} = \widehat{\mathcal{T}} \cap V$ with $\widehat{\mathcal{T}}$ a cellular tube around S.

2. If $r = 1$, *i.e.*, $v^{(1)} = (v)$, $\mathrm{PH}(v, S)$, $\mathrm{Sch}(v, S)$ and $\mathrm{Vir}(v, S)$ are called and denoted, respectively, the Poincaré–Hopf index $\mathrm{Ind}_{\mathrm{PH}}(v, S)$, the Schwartz index $\mathrm{Ind}_{\mathrm{Sch}}(v, S)$ and the virtual index $\mathrm{Ind}_{\mathrm{Vir}}(v, S)$ of the vector field v [71,111,148,149]. The corresponding Milnor class $\mu_0(V, S)$ is a number which will be discussed in Sect. 12.4.

12.3 Differential Geometric Point of View

In this section, we give a Lefschetz type formula for the Milnor classes at a nonsingular connected component S of the singular set of V under the assumption that V satisfies the Whitney condition along S. For the detailed proof, we refer to [31].

Let $\widehat{U}$ be a tubular neighborhood of S in M with C^∞ projection $\widehat{\rho} : \widehat{U} \to S$. We set $U = \widehat{U} \cap V$ and $U_0 = U \setminus S$ and denote by ρ and ρ_0, respectively, the restrictions of $\widehat{\rho}$ to U and U_0. From the Whitney condition, we see that the fibers of ρ are transverse to V and that S is a deformation retract of U with retraction ρ. We identify $\rho_0^*(N|_S)$ with N_{U_0}, and $\widehat{\rho}^*(N|_S)$ with $N|_{\widehat{U}}$. The bundle $T\widehat{\rho}$ of vectors in $T\widehat{U}$ tangent to the fibers of $\widehat{\rho}$ admits a complex structure, since it is C^∞ isomorphic with the normal bundle of the complex submanifold S in V. Let $\widehat{\mathcal{T}}$ be a (D)-cellular tube around S in $\widehat{U}$ and $\widehat{\mathcal{R}}$ a (D')-cellular tube in $\widehat{\mathcal{T}}$ as in Sect. 10.5.2. We set $\mathcal{T} = \widehat{\mathcal{T}} \cap V$ and $\mathcal{R} = \widehat{\mathcal{R}} \cap V$ as before.

Let s denote the complex dimension of S and let $v^{(r-1)}$ be an $(r-1)$-frame on the $2(s-r+1)$-skeleton $S \cap D^{(2q)}$ of S. In what follows, we set $\ell = s-r+1$. By the Schwartz construction, there exists a radial r-field $v_0^{(r)} = (v_0^{(r-1)}, v_0)$ on $\widehat{\mathcal{T}} \cap D^{(2q)}$ such that $v_0^{(r-1)}$ extends $v^{(r-1)}$. The radial vector field v_0 is tangent to U_0 and possibly has singularities in the barycenters of 2ℓ-cells in $S \cap D^{(2q)}$. We may assume that v_0 is tangent to the fibers of $\widehat{\rho}$ near $\partial\widehat{\mathcal{R}}$.

Let v be a vector field on $U_0 \cap D^{(2q)}$ which is nonsingular and tangent to the fibers of ρ in a neighborhood U_0' of $\partial\mathcal{R}$ so that $v^{(r)} = (v_0^{(r-1)}, v)$ is an r-frame on $U_0' \cap D^{(2q)}$. For example, the above v_0 has these properties.

For a point x in $S \cap D^{(2q)}$, let $\widehat{U}_x$ denote the fiber of $\widehat{\rho}$ at x and set $U_x = \widehat{U}_x \cap V$, which is the fiber of ρ at x. We also set $\mathcal{R}_x = \mathcal{R} \cap U_x$. The restriction of v to U_x determines the Schwartz index $\mathrm{Ind}_{\mathrm{Sch}}(v, S)$ and the virtual index $\mathrm{Ind}_{\mathrm{Vir}}(v, S)$ on U_x. By the Whitney condition, these indices do not depend on x.

Recall that we have the difference $d_S(v_0^{(r)}, v^{(r)})$ in $H_{2r-2}(S)$. We also have the difference $d(v_0, v)$, which is an integer, of v_0 and v as vector fields on U_x.

Lemma 12.3.1. *We have*

$$d_S(v_0^{(r)}, v^{(r)}) = d(v_0, v) \cdot c_{r-1}(S).$$

Proof. We consider the exact sequence of vector bundles on U_0:

$$0 \longrightarrow T\rho_0 \longrightarrow TU_0 \longrightarrow \rho_0^* TS \longrightarrow 0,$$

where $T\rho_0$ denotes the bundle of vectors in TU_0 tangent to the fibers of ρ_0. We may assume that $v_0^{(r)}$ and $v^{(r)}$ are r-frames on a neighborhood W of $U_0' \cap D^{(2q)}$. Let ∇_1^ρ and ∇_2^ρ be, respectively, v_0-trivial and v-trivial connections for $T\rho_0$

on W. Also let ∇^S be a $v_0^{(r-1)}$-trivial connection for TS on a neighborhood of $S \cap D^{(2q)}$. We take connections ∇_1 and ∇_2 for TU_0 so that $(\nabla_1^\rho, \nabla_1, \rho_0^*\nabla^S)$ and $(\nabla_2^\rho, \nabla_2, \rho_0^*\nabla^S)$ are both compatible with the above sequence. Thus ∇_1 is $v_0^{(r)}$-trivial and ∇_2 is $v^{(r)}$-trivial on W. By Lemma 10.5.1, the homology class $d_S(v_0^{(r)}, v^{(r)})$ is determined by

$$c^p(\nabla_1, \nabla_2) = \sum_{i+j=p} c^i(\nabla_1^\rho, \nabla_2^\rho) \cdot \rho_0^* c^j(\nabla^S). \tag{12.3.1}$$

We recall the commutative diagram

$$\begin{array}{ccc} H^{2q}(\widehat{U}, \widehat{U} \setminus S) & \xrightarrow[\hat{\rho}_*]{\sim} & H^{2\ell}(S) \\ \wr\downarrow A_M & & \wr\downarrow \\ H_{2r-2}(S) & \xrightarrow{=} & H_{2r-2}(S), \end{array} \tag{12.3.2}$$

where the first row is the inverse of the Thom isomorphism, given by integration along the fibers of $\widehat{\rho}$, and the second column is Poincaré duality. The dual of the first row in (12.3.2) gives an isomorphism

$$H_{2q}(\widehat{U}, \widehat{U} \setminus S) \xleftarrow{\sim} H_{2\ell}(S),$$

which shows that every relative $2q$-cycle γ (is homologous to a cycle which) fibers over a 2ℓ-cycle ζ of S. By the projection formula, we get from (12.3.1) (note that the rank of the bundle $T\rho_0$ is $n - s$):

$$\int_{\gamma \cap \partial \mathcal{R}} c^p(\nabla_1, \nabla_2) = \int_{\partial \mathcal{R}_x} c^{n-s}(\nabla_1^\rho, \nabla_2^\rho) \cdot \int_\zeta c^\ell(\nabla^S),$$

where x is a point in ζ. Noting that the first factor in the right hand side is $d(v_0, v)$, we proved the lemma, in view of (12.3.2).

Since $\mathrm{Ind}_{\mathrm{Sch}}(v_0^{(r)}, S) = c_{r-1}(S)$ and $\mathrm{Ind}_{\mathrm{Sch}}(v_0, x) = 1$, from Lemma 12.3.1, we have the following:

Theorem 12.3.3. *Let S be a nonsingular component of* $\mathrm{Sing}(V)$ *such that V satisfies the Whitney condition along S, then,*

$$\mathrm{Sch}(v^{(r)}, S) \;=\; \mathrm{Ind}_{\mathrm{Sch}}(v, x) \cdot c_{r-1}(S).$$

Now we wish to obtain a formula for the virtual class analogous to the one in Theorem 12.3.3. First, we consider the exact sequence of vector bundles on U_0:

$$0 \longrightarrow T\rho_0 \longrightarrow T\widehat{\rho}|_{U_0} \longrightarrow N_{U_0} \longrightarrow 0. \tag{12.3.4}$$

We compute the Chern classes $c^j(\tau_\rho)$ of the virtual bundle $\tau_\rho = (T\widehat{\rho} - N)|_U$ on U and will see that there is a canonical lifting $c^j_S(\tau_\rho)$ in $H^{2j}(U, U \setminus S)$, for $j > n - s = \operatorname{rank} T\rho_0$, of $c^j(\tau_\rho) \in H^{2j}(U)$. For this, we consider the covering $\mathcal{U}$ of $\widehat{U}$ consisting of U itself and a tubular neighborhood $\widehat{U}_0$ of U_0 and represent $c^j(\tau_{\rho_M})$, $\tau_{\rho_M} = T\widehat{\rho} - N$, as a Čech-de Rham cocycle on $\mathcal{U}$ (cf. [102, 156], here we use the notation in [156, Ch.II]).

Let ∇^ρ_0 be a connection for $T\rho_0$. Let ∇^N be a connection for $N|_S$ and take a connection $\nabla^{\hat{\rho}}_0$ for $T\widehat{\rho}|_{U_0}$ so that $(\nabla^\rho_0, \nabla^{\hat{\rho}}_0, \rho_0^*\nabla^N)$ is compatible with (12.3.4). Let $\hat{\nabla}^{\hat{\rho}}$ be a connection for $T\widehat{\rho}$ on $\widehat{U}$. We set $\nabla^{\widehat{\rho}\bullet} = (\hat{\nabla}^{\hat{\rho}}, \widehat{\rho}^*\nabla^N)$ and $\nabla^{\widehat{\rho}\bullet}_0 = (\nabla^{\widehat{\rho}}_0, \rho_0^*\nabla^N)$. Then $c^j(\tau_{\widehat{\rho}})$ is represented by a cocycle in $A^{2j}(\mathcal{U}) = A^{2j}(\widehat{U}_0) \oplus A^{2j}(\widehat{U}) \oplus A^{2j-1}(\widehat{U}_0)$, where $A^*(\ )$ denotes the space of differential forms on the relevant open set, given by

$$c^j(\nabla^\bullet_\star) = (c^j(\nabla^{\widehat{\rho}\bullet}_0), c^j(\nabla^{\widehat{\rho}\bullet}), c^j(\nabla^{\widehat{\rho}\bullet}_0, \nabla^{\widehat{\rho}\bullet})).$$

Note that, since $\widehat{U}_0$ retracts to U_0, it suffices to give forms on U_0. Since the family $(\nabla^\rho_0, \nabla^{\widehat{\rho}}_0, \rho_0^*\nabla^N)$ is compatible with (12.3.4), we have

$$c^j(\nabla^{\widehat{\rho}\bullet}_0) = c^j(\nabla^\rho_0),$$

which vanishes for $j > n-s$ by the rank reason. Thus, for $j > n-s$, the cocycle $c^j(\nabla^\bullet_\star)$ is in $A^{2j}(\mathcal{U}, \widehat{U}_0) = \{0\} \oplus A^{2j}(\widehat{U}) \oplus A^{2j-1}(\widehat{U}_0)$. Since the cohomology of $A^*(\mathcal{U}, \widehat{U}_0)$ is canonically isomorphic with $H^*(U, U \setminus S)$ [156, Ch.VI, 4], this cocycle defines a class, denoted $c^j_S(\tau_\rho)$, in $H^{2j}(U, U \setminus S)$, which is mapped to $c^j(\tau_\rho)$ by the canonical homomorphism $H^{2j}(U, U \setminus S) \to H^{2j}(U)$. The class $c^j_S(\tau_\rho)$ does not depend on the choices of various connections. It should be also noted that it does not depend on the frames we discussed earlier. Denoting by $A^{2i}(S)$ the space of $2i$-forms on S, we have the integration along the fibers of ρ [156, Ch.II, 5] $\rho_* : A^{2(n-s+i)}(\mathcal{U}, \widehat{U}_0) \to A^{2i}(S)$, which commutes with the differentials and induces a map on the cohomology level :

$$\rho_* : H^{2(n-s+i)}(U, U \setminus S) \longrightarrow H^{2i}(S).$$

On the cocycle level, ρ_* assigns to $c^{n-s+i}(\nabla^\bullet_\star)$, $i > 0$, the $2i$-form α^i on S given by

$$\alpha^i = \rho_* c^{n-s+i}(\nabla^{\widehat{\rho}\bullet}_M) + (\partial\rho)_* c^{n-s+i}(\nabla^{\widehat{\rho}\bullet}_M, \nabla^{\widehat{\rho}\bullet}_0), \tag{12.3.5}$$

where ρ_* and $(\partial\rho)_*$ denote the integration along the fibers of $\rho|_{\mathcal{R}}$ and $\rho|_{\partial\mathcal{R}}$.

We note that, in the following formulas, the classes $\rho_* c^{n-s+i}_S(\tau_\rho)$ for $i = 1, \ldots, k-1$ are involved and they do not appear if $k = 1$ (*i.e.*, V is a hypersurface). We denote by $[\ \ ]^i$ the component of degree $2i$ of the relevant cohomology class.

Theorem 12.3.6. *With the hypotheses of Theorem 12.3.3, we have*

$$\mathrm{Vir}(v^{(r)}, S) = \Bigg[\Bigg(\mathrm{Ind}_{\mathrm{Vir}}(v,x)\cdot(c^*(N)-c^k(N)) + \mathrm{Ind}_{\mathrm{Sch}}(v,x)\cdot c^k(N)$$

$$+\sum_{j=1}^{k-1}\sum_{i=1}^{j} c^{j-i}(N)\cdot \rho_* c_S^{n-s+i}(\tau_\rho)\Bigg)\cdot c^*(N)^{-1}\cdot c^*(S)\Bigg]^{\ell} \frown [S].$$

From Theorems 12.3.3 and 12.3.6, we get the following Lefschetz type formula for the Milnor class.

Corollary 12.3.1. *Let S be a nonsingular connected component of* $\mathrm{Sing}(V)$ *such that V satisfies the Whitney condition along S. Then*

$$\mu_*(V,S) = \Bigg((-1)^s \mu(V\cap H, x)\cdot(c^*(N)-c^k(N))$$

$$+(-1)^n \sum_{j=1}^{k-1}\sum_{i=1}^{j} c^{j-i}(N)\cdot \rho_* c_S^{n-s+i}(\tau_\rho)\Bigg)\cdot c^*(N)^{-1}\cdot c^*(S)\frown [S],$$

where H denotes an $(m-s)$-dimensional plane transverse to S in M. In particular, if $k=1$,

$$\mu_*(V,S) = (-1)^s \mu(V\cap H, x)\cdot c^*(N)^{-1}\cdot c^*(S)\frown [S].$$

Also, for arbitrary k,

$$\mu_s(V,S) = (-1)^s \mu(V\cap H, x)\cdot [S].$$

Remark 12.3.1. 1. In [131], the Milnor class of a hypersurface V is defined by $\mu_*(V) = (-1)^n\big(c_*(V) - c^*(\tau_V)\frown [V]\big)$ and a formula for this is given as a sum of the contributions from the strata of a stratification of V. This result was obtained earlier for the Milnor number $\mu_0(V)$ in [130] and, for the Milnor class, it was conjectured in [169]. If the stratum is a nonsingular component of $\mathrm{Sing}(V)$, its contribution coincides with the one given in the second formula above.

2. In fact, the formulas in Corollary 12.3.1 hold under an assumption weaker than the Whitney condition. Namely, we only need that there is a Whitney stratification of M compatible with V and S such that the $2(\ell-r)$-skeleton $S\cap D^{2(q-1)}$ of S is in the top dimensional stratum of S. Accordingly, under this assumption, we have a formula for $\mu_r(V,S)$ taking the terms of corresponding dimension in the above formulas (see [31]).

12.4 Generalized Milnor Number

As in the previous sections, let $V \subset M$ be defined by a holomorphic section of a vector bundle of rank k and let S be a connected component of $\mathrm{Sing}(V)$.

Definition 12.4.1. The *generalized Milnor number* $\mu(V,S)$ of V at S is defined as

$$\mu(V,S) = (-1)^{n+1}\big(\mathrm{Ind}_{\mathrm{Sch}}(v,S) - \mathrm{Ind}_{\mathrm{Vir}}(v,S)\big),$$

where v is a vector field on a neighborhood U of S in V, nonsingular on $U \setminus S$.

This definition does not depend on the choice of the vector field v and is equal to $\mu_0(V,S)$ in Definition 12.2.1. If (V,a) is an isolated complete intersection singularity germ, for a radial vector field v_0, $\mathrm{Ind}_{\mathrm{Sch}}(v_0,a) = 1$ and $\mathrm{Ind}_{\mathrm{Vir}}(v_0,a) = \chi(\mathbf{F})$, where $\mathbf{F}$ denotes the Milnor fiber. Thus the above Milnor number coincides with the usual one in [79, 116, 121].

We recall that the classical Milnor number of an isolated singular point [121] has been generalized to the case of nonisolated hypersurface singularities by A. Parusiński [127] in the following way. Recall that a hypersurface V in M is always defined by a holomorphic section s of a line bundle N over M. There is a canonical vector bundle homomorphism $\pi : TM|_V \to N|_V$ which extends the one in (11.4.1). Note that $\mathrm{Sing}(V)$ coincides with the set of points in V where π fails to be surjective. Now let ∇' be a connection for N of type $(1,0)$. This means that in the decomposition $\nabla' = \nabla^{(1,0)} + \nabla^{(0,1)}$ of ∇' into the $(1,0)$ and $(0,1)$ components, we have $\nabla^{(0,1)} = \bar\partial$. Since s is holomorphic, we have $\nabla' s = \nabla^{(1,0)} s$, which is a C^∞ section t of $T^*M \otimes N$. Write $\tilde\pi : TM \to N$ the corresponding bundle homomorphism. Let S be a compact component of $\mathrm{Sing}(V)$ and $\widehat{U}$ a neighborhood of S in M disjoint from the other components. It is shown in [127] that S coincides with a connected component of the zero set of t. Then Parusiński defines the Milnor number $\mu_S(V)$ to be the intersection number in $\widehat{U}$ of the section t of $T^*M \otimes N$ with the zero section. We refer to [31] for the proof of the following

Theorem 12.4.1. *For a hypersurface V, we have*

$$\mu_S(V) = \mu(V,S).$$

Chapter 13
Characteristic Classes of Coherent Sheaves on Singular Varieties

Abstract As we have seen along this book, for a singular variety V, there are several definitions of Chern classes, the Mather class, the Schwartz–MacPherson class, the Fulton–Johnson class and so forth. They are in the homology of V and, if V is nonsingular, they all reduce to the Poincaré dual of the Chern class $c^*(TV)$ of the tangent bundle TV of V. On the other hand, for a coherent sheaf $\mathcal{F}$ on V, the (cohomology) Chern character $\mathrm{ch}^*(\mathcal{F})$ or the Chern class $c^*(\mathcal{F})$ makes sense if either V is nonsingular or $\mathcal{F}$ is locally free. In this chapter, we propose a definition of the homology Chern character $\mathrm{ch}_*(\mathcal{F})$ or the Chern class $c_*(\mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on a possibly singular variety V. In this direction, the homology Chern character or the Chern class is defined in [140] (see also [100]) using the Nash type modification of V relative to the linear space associated to the coherent sheaf $\mathcal{F}$. Also, the homology Todd class $\tau(\mathcal{F})$ is introduced in [15] to describe their Riemann-Roch theorem. Our class is closely related to the latter.

The variety V we consider in this chapter is a local complete intersection defined by a section of a holomorphic vector bundle over the ambient complex manifold M. If $\mathcal{F}$ is a locally free sheaf on V, then the class $\mathrm{ch}_*(\mathcal{F})$ coincides with the image of $\mathrm{ch}^*(\mathcal{F})$ by the Poincaré homomorphism $H_*(V) \to H^*(V)$. This fact follows from the Riemann-Roch theorem for the embedding of V into M, which we prove at the level of Čech-de Rham cocycles. We also compute the Chern character and the Chern class of the tangent sheaf of V, in the case V has only isolated singularities.

13.1 Local Chern Classes and Characters in the Čech-de Rham Cohomology

Let M be a C^∞ manifold and E a C^∞ complex vector bundle E over M. For a connection ∇ of E, let K denote its curvature and set $A = (\sqrt{-1}/2\pi)K$. We represent K locally by a curvature matrix. Recall that the total Chern form is defined by (cf. Sect. 1.4)

$$c^*(\nabla) = \det(I + A).$$

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We further define the Chern character form and the Todd form by

$$\mathrm{ch}^*(\nabla) = \mathrm{tr}(e^A),$$
$$\mathrm{td}(\nabla) = \det\left(\frac{A}{I - e^{-A}}\right).$$

Note that $I - e^{-A}$ is divisible by A and the result is invertible so that

$$\mathrm{td}^{-1}(\nabla) = \det\left(\frac{I - e^{-A}}{A}\right)$$

also makes sense. If we set $s^i(\nabla) = \mathrm{tr}(A)^i$, the homogeneous piece of degree $2i$ in $\mathrm{tr}(A)$, then it is a closed $2i$-form on M. Denoting by ℓ the rank of E, we have

$$c^*(\nabla) = 1 + \sum_{i=1}^{\ell} c^i(\nabla) \qquad \text{and} \qquad \mathrm{ch}^*(\nabla) = \ell + \sum_{i\geq 1} \frac{s^i(\nabla)}{i!}.$$

The forms $c^i = c^i(\nabla)$ and $s^i = s^i(\nabla)$ are related by Newton's formula:

$$s^i - c^1 s^{i-1} + c^2 s^{i-2} - \cdots + (-1)^i i\, c^i = 0, \qquad i \geq 1. \tag{13.1.1}$$

The class of $\mathrm{ch}^*(\nabla)$ in $H^*(M, \mathbb{C})$ is the (cohomology) Chern character $\mathrm{ch}^*(E)$ of E. Each homogeneous piece of $\mathrm{td}(\nabla)$ is also closed and the class of $\mathrm{td}(\nabla)$ in $H^*(M, \mathbb{C})$ is the Todd class $\mathrm{td}(E)$ of E. Note that the constant term in $\mathrm{td}(\nabla)$ is 1 and that $\mathrm{td}(\nabla)$ can be expressed as a series (in fact a polynomial) in $c^i(\nabla)$. We have the following fundamental formula [80, III, Corollary 5.4]:

$$\sum_{i=0}^{\ell} (-1)^i \mathrm{ch}^*(\Lambda^i \nabla^*) = \mathrm{td}^{-1}(\nabla) \cdot c^{\ell}(\nabla), \tag{13.1.2}$$

where ∇^* denotes the connection for E^* dual to ∇ and $\Lambda^i \nabla^*$ the connection for $\Lambda^i E^*$ induced by ∇^*. Here we set $\Lambda^0 E^* = M \times \mathbb{C}$ (the trivial line bundle) and $\Lambda^0 \nabla^* = d$. See, e.g., [84, Theorem 10.1.1] for the above formula in cohomology.

Let $\xi = \sum_{i=0}^{q} (-1)^i E_i$ be a virtual bundle and $\nabla^\bullet = (\nabla^{(q)}, \ldots, \nabla^{(0)})$ a family of connections, each $\nabla^{(i)}$ being a connection for E_i. We set

$$c^*(\nabla^\bullet) = \prod_{i=0}^{q} c^*(\nabla^{(i)})^{\varepsilon(i)} \qquad \text{and} \qquad \mathrm{ch}^*(\nabla^\bullet) = \sum_{i=0}^{q} (-1)^i \mathrm{ch}^*(\nabla^{(i)}),$$

where $\varepsilon(i) = (-1)^i$. If we denote by $c^i = c^i(\nabla^\bullet)$ and $s^i/i! = s^i(\nabla^\bullet)/i!$ the homogeneous pieces of degree $2i$ in $c^*(\nabla^\bullet)$ and $\mathrm{ch}^*(\nabla^\bullet)$, respectively, they are again related by (13.1.1). More generally, if $\varphi = \varphi(c^1, c^2, \dots)$ is a series in c^i (we call such a series a symmetric series), we may define a form $\varphi(\nabla^\bullet)$ (cf. Sect. 5.2). It is a closed form and its class $\varphi(\xi)$ in the cohomology ring $H^*(M;\mathbb{C})$ is the characteristic class of ξ with respect to φ. Suppose further that we have two families of connections $\nabla^\bullet_\nu = (\nabla^{(q)}_\nu, \dots, \nabla^{(0)}_\nu)$, $\nu = 0, 1$, for ξ. Then, we have the "difference form" $\varphi(\nabla^\bullet_0, \nabla^\bullet_1)$ satisfying (5.2.1).

Now we state a lemma which will be used to describe explicitly the difference between the cocycle for the product of two symmetries series and the product of cocycles for these series. For the proof we refer to [158, Lemma 1.5] Note that $\varphi\psi(\nabla^\bullet) = \varphi(\nabla^\bullet)\cdot\psi(\nabla^\bullet)$, for symmetric series φ and ψ and a family of connections $\nabla^\bullet$.

Lemma 13.1.1. *In the above situation, for two symmetric series φ and ψ, we have*

$$\varphi\psi(\nabla^\bullet_0, \nabla^\bullet_1) = \varphi(\nabla^\bullet_0)\cdot\psi(\nabla^\bullet_0, \nabla^\bullet_1) + \varphi(\nabla^\bullet_0, \nabla^\bullet_1)\cdot\psi(\nabla^\bullet_1) - d\,\tau_{01},$$

where

$$\tau_{01} = \pi_*(\varphi(\pi^*\nabla^\bullet_0, \tilde{\nabla}^\bullet)\cdot d\psi(\pi^*\nabla^\bullet_1, \tilde{\nabla}^\bullet)).$$

Let M be as above and let $\mathcal{U}$ be an open covering of M consisting of two open sets U_0 and U_1.

If $\xi = \sum_{i=0}^q (-1)^i E_i$ is a virtual bundle, we take a family of connections $\nabla^\bullet_\nu = (\nabla^{(q)}_\nu, \dots, \nabla^{(0)}_\nu)$ for ξ on each U_ν, $\nu = 0, 1$. Recall that, for the collection $\nabla^\bullet_\star = (\nabla^\bullet_0, \nabla^\bullet_1)$ and a symmetric series φ, we have the cochain $\varphi(\nabla^\bullet_\star)$ as defined by (5.2.4) in $A^*(\mathcal{U})$. It is a cocycle and defines a class $[\varphi(\nabla^\bullet_\star)]$ in $H^*_D(\mathcal{U})$. It corresponds to the class $\varphi(\xi)$ under the isomorphism $H^*_D(\mathcal{U}) \simeq H^*(M;\mathbb{C})$.

From Lemma 13.1.1, we have the following:

Proposition 13.1.1. *For two symmetric series φ and ψ, we have, in $A^*(\mathcal{U})$,*

$$\varphi\psi(\nabla^\bullet_\star) = \varphi(\nabla^\bullet_\star) \smile \psi(\nabla^\bullet_\star) + D\tau,$$

where $\tau = (0, 0, \tau_{01})$ with τ_{01} a form on U_{01} as given in Lemma 13.1.1.

Now we discuss the localization theory of characteristic classes as considered above.

To describe these, let M be as above and let S be a closed set in M. Letting $U_0 = M \setminus S$ and U_1 a neighborhood of S in M, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Note that the cup product of a cochain in $A^*(\mathcal{U})$ and a cochain in $A^*(\mathcal{U}, U_0)$ is in $A^*(\mathcal{U}, U_0)$ and this induces a natural $H^*(M;\mathbb{C})$-module structure on $H^*(M, M \setminus S;\mathbb{C})$.

Remark 13.1.1. In the situation of Proposition 13.1.1, if $\psi(\nabla_\star^\bullet)$ is in $A^*(\mathcal{U}, U_0)$, *i.e.*, if $\psi(\nabla_0^\bullet) = 0$, then so is $\varphi\psi(\nabla_\star^\bullet)$, since $\varphi\psi(\nabla_0^\bullet) = \varphi(\nabla_0^\bullet) \cdot \psi(\nabla_0^\bullet)$. The proposition shows that the class $\varphi\psi(\xi)$ coincides with $\varphi(\xi) \smile \psi(\xi)$ in $H^*(M, M \setminus S; \mathbb{C})$, since τ is also in $A^*(\mathcal{U}, U_0)$.

Now we consider the localization of the Chern classes of a virtual bundle by exactness.

Let

$$0 \longrightarrow E_q \xrightarrow{h_q} \cdots \xrightarrow{h_1} E_0 \longrightarrow 0 \tag{13.1.3}$$

be a complex of C^∞ complex vector bundles over M which is exact on U_0. Then we will see below that, for each $i > 0$, there is a canonical localization $c_S^i(\xi)$ in $H^{2i}(M, M \setminus S; \mathbb{C})$ of the Chern class $c^i(\xi)$ in $H^{2i}(M; \mathbb{C})$ of the virtual bundle $\xi = \sum_{i=0}^q (-1)^i E_i$. For this, we recall the following "vanishing theorem" ([14, Lemma (4.22)]) for a family of connections "compatible" with the sequence (13.1.3) (cf. (5.2.3)):

Lemma 13.1.2. *If $\nabla_0^\bullet$ is a family of connections on U_0 compatible with (13.1.3), then, for each $i > 0$,*

$$c^i(\nabla_0^\bullet) = 0.$$

In fact, the above holds for the difference form of a finite number of families of connections compatible with (13.1.3) on U_0. For a symmetric series φ without constant term, we also have a similar vanishing $\varphi(\nabla_0^\bullet) = 0$.

Let $\nabla_0^\bullet$ be a family of connections compatible with (13.1.3) on U_0 and $\nabla_1^\bullet$ an arbitrary family of connections for $\xi = \sum_{i=0}^q (-1)^i E_i$ on U_1. Then the class $c^i(\xi)$ is represented by the cocycle

$$c^i(\nabla_\star^\bullet) = (c^i(\nabla_0^\bullet),\, c^i(\nabla_1^\bullet),\, c^i(\nabla_0^\bullet, \nabla_1^\bullet))$$

in $A^{2i}(\mathcal{U})$. By Lemma 13.1.2, we have $c^i(\nabla_0^\bullet) = 0$ and thus the cocycle is in $A^{2i}(\mathcal{U}, U_0)$ and it defines a class $c_S^i(\xi)$ in $H^{2i}(M, M \setminus S; \mathbb{C})$. It is sent to $c^i(\xi)$ by the canonical homomorphism j^*. It is not difficult to see that the class $c_S^i(\xi)$ does not depend on the choice of the family of connections $\nabla_0^\bullet$ compatible with (13.1.3) or on the choice of the family of connections $\nabla_1^\bullet$.

If φ is a symmetric series without constant term, we may also define the localized class $\varphi_S(\xi)$ of $\varphi(\xi)$. In particular, noting that the alternating sum of the ranks of E_i is zero, if $M \setminus S \neq \emptyset$, we have the localized Chern character $\mathrm{ch}_S^*(\xi)$ in the relative cohomology $H^*(M, M \setminus S; \mathbb{C})$, which is sent to $\mathrm{ch}^*(\xi)$ by the homomorphism j^*. It is the class of the cocycle

$$\mathrm{ch}^*(\nabla_\star^\bullet) = (0,\, \mathrm{ch}^*(\nabla_1^\bullet),\, \mathrm{ch}^*(\nabla_0^\bullet, \nabla_1^\bullet))$$

in $A^*(\mathcal{U}, U_0)$.

Let E be another vector bundle over M and ∇ a connection for E on M. Then its Chern character $\mathrm{ch}^*(E)$ is the class of the cocycle

$$\mathrm{ch}^*(\nabla) = (\mathrm{ch}^*(\nabla), \mathrm{ch}^*(\nabla), 0)$$

in $A^*(\mathcal{U})$. The complex

$$0 \longrightarrow E \otimes E_q \longrightarrow \cdots \longrightarrow E \otimes E_0 \longrightarrow 0$$

is exact on U_0 and the family $\nabla \otimes \nabla_0^\bullet = (\nabla \otimes \nabla_0^{(q)}, \ldots, \nabla \otimes \nabla_0^{(0)})$ of connections is compatible with the above sequence on U_0. We set $E \otimes \xi = \sum_{i=0}^q (-1)^i E \otimes E_i$ and let $\nabla \otimes \nabla_1^\bullet$ denote the family $(\nabla \otimes \nabla_1^{(q)}, \ldots, \nabla \otimes \nabla_1^{(0)})$. Then $\mathrm{ch}^*(E \otimes \xi)$ is the class of the cocycle

$$\mathrm{ch}^*(\nabla \otimes \nabla_\star^\bullet) = (0, \mathrm{ch}^*(\nabla \otimes \nabla_1^\bullet), \mathrm{ch}^*(\nabla \otimes \nabla_0^\bullet, \nabla \otimes \nabla_1^\bullet)).$$

We have

$$\mathrm{ch}^*(\nabla \otimes \nabla_1^\bullet) = \mathrm{ch}^*(\nabla) \cdot \mathrm{ch}^*(\nabla_1^\bullet),$$
$$\mathrm{ch}^*(\nabla \otimes \nabla_0^\bullet, \nabla \otimes \nabla_1^\bullet) = \mathrm{ch}^*(\nabla) \cdot \mathrm{ch}^*(\nabla_0^\bullet, \nabla_1^\bullet).$$

Hence, recalling the definition of the cup product, we have

$$\mathrm{ch}^*(\nabla \otimes \nabla_\star^\bullet) = \mathrm{ch}^*(\nabla) \smile \mathrm{ch}^*(\nabla_\star^\bullet) \tag{13.1.4}$$

in $A^*(\mathcal{U}, U_0)$. In particular, we have

$$\mathrm{ch}_S^*(E \otimes \xi) = \mathrm{ch}^*(E) \smile \mathrm{ch}_S^*(\xi).$$

Remark 13.1.2. The local Chern characters defined as above have all the necessary properties and should coincide with the ones in [86]. Hence they are in the cohomology $H^*(M, M \setminus S; \mathbb{Q})$ with $\mathbb{Q}$ coefficients. Also, the local Chern classes above are in the image of $H^*(M, M \setminus S; \mathbb{Z}) \to H^*(M, M \setminus S; \mathbb{C})$. See also [15] for local Chern characters.

Now let M be a complex manifold and denote by $\mathcal{O}_M$ and $\mathcal{A}_M$, respectively, the sheaves of germs of holomorphic functions and of real analytic functions on M. If U is a relatively compact open set in M and if $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module, there is a complex of real analytic vector bundles on U as (1.8) such that at the sheaf level

$$0 \longrightarrow \mathcal{A}_U(E_q) \longrightarrow \cdots \longrightarrow \mathcal{A}_U(E_0) \longrightarrow \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{F} \longrightarrow 0 \tag{13.1.5}$$

is exact [11]. We call such a sequence a resolution of $\mathcal{F}$ by vector bundles. We define the Chern character $\mathrm{ch}^*(\mathcal{F})$ of $\mathcal{F}$ by $\mathrm{ch}^*(\mathcal{F}) = \mathrm{ch}^*(\xi)$, $\xi = \sum_{i=0}^q (-1)^i E_i$. Then it does not depend on the choice of the resolution. If we denote by S the support of $\mathcal{F}$, then it is an analytic set in U and on $U \setminus S$, the sequence (13.1.3) is exact. Thus we have the localized Chern character $\mathrm{ch}_S^*(\mathcal{F})$ in $H^*(U, U \setminus S; \mathbb{C})$. If E is a vector bundle over U, the characteristic classes of $E \otimes \mathcal{F}$ are those of $E \otimes \xi$. Hence, from (13.1.4), we have

$$\mathrm{ch}_S^*(E \otimes \mathcal{F}) = \mathrm{ch}^*(E) \smile \mathrm{ch}_S^*(\mathcal{F}). \tag{13.1.6}$$

Note that the above equality also holds if we replace E by a virtual bundle over U.

13.2 Thom Class

Let M be a complex manifold of dimension $m = n + k$ and V a compact analytic subvariety of pure dimension n in M. We denote by i the embedding $V \hookrightarrow M$. If $V = \bigcup_{i=1}^r V_i$ is the irreducible decomposition of V, we set $[V] = \sum_{i=1}^r [V_i]$ in $H_{2n}(V)$.

Recall that (10.4.6) we have the Thom homomorphism

$$T : H^p(V) \longrightarrow H^{p+2k}(M, M \setminus V).$$

For the class [1] in $H^0(V)$, we denote $T([1])$ in $H^{2k}(M, M \setminus V)$ by Ψ_V, and call it the *Thom class* of V in M.

Let U be a regular neighborhood of V in M with continuous retraction $\rho : U \to V$. We have, by excision, $H^*(M, M \setminus V) \simeq H^*(U, U \setminus V)$. Note that for σ in $H^*(U)$ and τ in $H^*(U, U \setminus V)$, we have

$$A(\sigma \smile \tau) = i^*\sigma \frown A(\tau),$$

where A denotes the Alexander isomorphism (see Sects. 1.2 and 10.4)

$$A : H^*(U, U \setminus V) \longrightarrow H_*(V).$$

Hence the Thom homomorphism T is given, for a class α in $H^p(V)$, by

$$T(\alpha) = \rho^*(\alpha) \smile \Psi_V. \tag{13.2.1}$$

We also have the Gysin homomorphism (10.4.3)

$$i_* : H^p(V) \to H^{p+2k}(M),$$

which satisfy $i_* = j^* \circ T$. Note that, if M is compact, we have the commutative diagram (cf. Sect. 10.4)

$$\begin{array}{ccccc} H^p(V) & \xrightarrow{T} & H^{p+2k}(M, M \setminus V) & \xrightarrow{j^*} & H^{p+2k}(M) \\ \big\downarrow P_V & & \wr\big\downarrow A & & \wr\big\downarrow P_M \\ H_{2n-p}(V) & \xrightarrow{=} & H_{2n-p}(V) & \xrightarrow{i_*} & H_{2n-p}(M). \end{array}$$

In this and the subsequent sections, we consider the following two cases:
(i) V is nonsingular,
(ii) V is a local complete intersection defined by a section (see Sect. 5.1).

First, suppose V is nonsingular and let $p : N_V \to V$ be the normal bundle of V in M. In this case, P and T are isomorphisms. We may take as U above a tubular neighborhood so that ρ is C^∞. Then $\rho : U \to V$ is isomorphic with $p : W \to V$ for a neighborhood W of the zero section in N_V, which we identify with V. The bundle $\rho^* N_V$ is also isomorphic with $p^* N_V$. Thus we have an isomorphism

$$H^*(M, M \setminus V) \simeq H^*(N_V, N_V \setminus V).$$

The Thom class Ψ_V of V corresponds to the Thom class Ψ_{N_V} of the bundle N_V under this isomorphism and the Thom homomorphism corresponds to the Thom isomorphism $T_{N_V} : H^p(V) \xrightarrow{\sim} H^{p+2k}(N_V, N_V \setminus V)$. Note that, if we denote by s_Δ the diagonal section of the bundle $p^* N_V$ over N_V, its zero set is V and we have ([156, Ch.III, Theorem 4.4])

$$\Psi_{N_V} = c^k(p^* N_V, s_\Delta).$$

Second, recall that a subvariety V of codimension k in M is a local complete intersection (abbreviated as LCI) in M if the ideal sheaf $\mathcal{I}_V$ in $\mathcal{O}_M$ of functions vanishing on V is locally generated by k functions. In this case, the normal sheaf $\mathcal{N}_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$ is a locally free $\mathcal{O}_V$-module, $\mathcal{O}_V = \mathcal{O}_M/\mathcal{I}_V$. We denote by N_V the associated vector bundle.

If V is an LCI defined by a section s of a vector bundle N of rank k over M (cf. Sect. 5.1), $N_V = N|_V$ and we have (cf. [157], [161])

$$\Psi_V = c^k(N, s). \tag{13.2.2}$$

13.3 Riemann-Roch Theorem for Embeddings

Let V be a compact subvariety in a complex manifold M, which is either of type (i) or (ii) in the previous section. Let U be a regular neighborhood of V in M with a continuous retraction $\rho : U \to V$. In the case (ii), suppose V is defined by a section s of a vector bundle N over M. In the case (i), (M, V)

is C^∞ diffeomorphic with (N_V, V) and, in the latter, V is defined by the diagonal section s_Δ of the bundle p^*N_V over N_V. In what follows we write N_V by M anew and set $N = p^*N_V$ and $s = s_\Delta$. Thus in either case we may express the Thom class Ψ_V as (13.2.2). In the case (i), we may take as U a tubular neighborhood and we may assume that ρ is the restriction of p to U.

Let $U_0 = M \setminus V$ and U_1 a neighborhood of V as before. Also, let ∇_0 be an s-trivial connection for N on U_0 and ∇_1 an arbitrary connection for N on U_1. We consider the vector bundle $N \times \mathbb{R}$ over $U_{01} \times \mathbb{R}$ and let $\tilde{\nabla}$ be the connection for it given by $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$. Let $\Lambda^\bullet \nabla_\nu^*$ denote the family of connections $(\Lambda^k \nabla_\nu^*, \dots, \Lambda^0 \nabla_\nu^*)$ on U_ν, for $\nu = 0, 1$. Also denote by $\Lambda^\bullet \tilde{\nabla}^*$ the family $(\Lambda^k \tilde{\nabla}^*, \dots, \Lambda^0 \tilde{\nabla}^*)$. Let $\pi : U_{01} \times [0,1] \to U_{01}$ be the projection. Recall that, in $A^*(\mathcal{U})$,

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^*) = (\mathrm{ch}^*(\Lambda^\bullet \nabla_0^*),\, \mathrm{ch}^*(\Lambda^\bullet \nabla_1^*),\, \mathrm{ch}^*(\Lambda^\bullet \nabla_0^*, \Lambda^\bullet \nabla_1^*))$$

whose class in $H^*(M;\mathbb{C})$ is $\mathrm{ch}^*(\lambda_{N^*})$, $\lambda_{N^*} = \sum_{i=0}^k (-1)^i \Lambda^i N^*$.

Theorem 13.3.1. *The cocycle* $\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^*)$ *is in* $A^*(\mathcal{U}, U_0)$ *and is given by*

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^*) = \mathrm{td}^{-1}(\nabla_\star) \smile c^k(\nabla_\star) + D\tau,$$

where $\tau = (0, 0, \tau_{01})$, $\tau_{01} = \pi_*(\mathrm{td}^{-1}(\pi^*\nabla_0, \tilde{\nabla}) \cdot d\, c^k(\pi^*\nabla_1, \tilde{\nabla}))$.

Proof. By (13.1.2), we have

$$\begin{aligned}
\mathrm{ch}^*(\Lambda^\bullet \nabla_0^*) &= \mathrm{td}^{-1}(\nabla_0) \cdot c^k(\nabla_0) = 0, \\
\mathrm{ch}^*(\Lambda^\bullet \nabla_1^*) &= \mathrm{td}^{-1}(\nabla_1) \cdot c^k(\nabla_1), \\
\mathrm{ch}^*(\Lambda^\bullet \nabla_0^*, \Lambda^\bullet \nabla_1^*) &= \pi_* \mathrm{ch}^*(\Lambda^\bullet \tilde{\nabla}^*) \\
&= \pi_*(\mathrm{td}^{-1}(\tilde{\nabla}) \cdot c^k(\tilde{\nabla})) = (\mathrm{td}^{-1} \cdot c^k)(\nabla_0, \nabla_1).
\end{aligned}$$

Hence we see that

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^*) = (\mathrm{td}^{-1} \cdot c^k)(\nabla_\star)$$

and the theorem follows from Proposition 13.1.1 (see also Remark 13.1.1).

Remark 13.3.1. Consider the Koszul complex associated to s [59, B.3]:

$$0 \longrightarrow \Lambda^k N^* \longrightarrow \cdots \longrightarrow \Lambda^1 N^* \longrightarrow \Lambda^0 N^* \longrightarrow 0, \tag{13.3.2}$$

which is exact on $U_0 = M \setminus V$. It is not difficult to see that the family $\Lambda^\bullet \nabla_0^*$ is compatible with the sequence (13.3.2) on U_0. The fact that $\mathrm{ch}^*(\Lambda^\bullet \nabla_0^*) = 0$ also follows from this (cf. Lemma 13.1.2).

Let $\mathcal{F}$ be a coherent $\mathcal{O}_V$-module. The direct image $i_!\mathcal{F}$ is a coherent $\mathcal{O}_M$-module, which is simply $\mathcal{F}$ extended by zero on $M \setminus V$, and thus we have the localized Chern character $\mathrm{ch}_V^*(i_!\mathcal{F})$ in $H^*(M, M \setminus V; \mathbb{C})$.

In the case (i), we take a resolution of $\mathcal{F}$ of the form (13.1.5) on V. Then we have $\mathrm{ch}^*(\mathcal{F}) = \mathrm{ch}^*(\xi)$, $\xi = \sum_{i=0}^{q}(-1)^i E_i$. Let $\nabla^{(i)}$ be a connection for E_i, $i = 0, \dots, q$, and denote by $\nabla^{\mathcal{F}}$ the family of connections $(\rho^*\nabla^{(0)}, \dots, \rho^*\nabla^{(q)})$, for the virtual bundle $\rho^*\xi$ over U.

In the case (ii), we assume that $\mathcal{F}$ is locally free and thus $\mathcal{F} = \mathcal{O}_V(F)$ for some vector bundle F over V. Since the classification of continuous vector bundles and that of C^∞ vector bundles coincide over paracompact manifold s, we may assume that ρ^*F is a C^∞ vector bundle and let $\nabla^{\mathcal{F}}$ be a connection for ρ^*F on U.

In either case, let $\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star)$ denote the cocycle

$$\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) = (\mathrm{ch}^*(\nabla^{\mathcal{F}}), \mathrm{ch}^*(\nabla^{\mathcal{F}}), 0)$$

in $A^*(\mathcal{U})|_U$, whose class in $H^*(U;\mathbb{C})$ is $\rho^*\mathrm{ch}^*(\mathcal{F})$.

Corollary 13.3.1. *In the above situation, we have*

$$\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) \smile \mathrm{ch}^*(\Lambda^\bullet\nabla^*_\star) = \mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) \smile \mathrm{td}^{-1}(\nabla_\star) \smile c^k(\nabla_\star) + D(\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) \smile \tau)$$

in $A^*(\mathcal{U}, U_0)|_U$.

Corollary 13.3.2. *Let V be a compact subvariety in M and $\mathcal{F}$ a coherent $\mathcal{O}_V$-module. We have the following formulas in either one of the cases:*
(i) *V is nonsingular,*
(ii) *V is an LCI defined by a section and $\mathcal{F}$ is locally free.*

$$\begin{aligned} \mathrm{ch}^*_V(i_!\mathcal{F}) &= T(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) \qquad \text{in} \quad H^*(M, M \setminus V; \mathbb{C}), \\ \mathrm{ch}^*(i_!\mathcal{F}) &= i_*(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) \qquad \text{in} \quad H^*(M; \mathbb{C}). \end{aligned}$$

Proof. The Koszul complex (13.3.2) gives a locally free resolution of $i_!\mathcal{O}_V$:

$$0 \longrightarrow \mathcal{O}_M(\Lambda^k N^*) \longrightarrow \cdots \longrightarrow \mathcal{O}_M(\Lambda^0 N^*) \longrightarrow i_!\mathcal{O}_V \longrightarrow 0.$$

If we compute the local class $\mathrm{ch}^*_V(i_!\mathcal{O}_V)$ using this resolution, we see that it is represented by $\mathrm{ch}^*(\Lambda^\bullet\nabla^*_\star)$. We have, by (13.1.6),

$$\mathrm{ch}^*_V(i_!\mathcal{F}) = \begin{cases} \mathrm{ch}^*(\rho^*\xi \otimes i_!\mathcal{O}_V) = \mathrm{ch}^*(\rho^*\xi) \smile \mathrm{ch}^*_V(i_!\mathcal{O}_V), & \text{in the case (i)} \\ \mathrm{ch}^*(\rho^*F \otimes i_!\mathcal{O}_V) = \mathrm{ch}^*(\rho^*F) \smile \mathrm{ch}^*_V(i_!\mathcal{O}_V), & \text{in the case (ii).} \end{cases}$$

Recall that either $\mathrm{ch}^*(\rho^*\xi)$ or $\mathrm{ch}^*(\rho^*F)$ is represented by $\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star)$. Recalling also that $N|_U \simeq \rho^*N_V$ and $c^k(N, s) = \Psi_V$ (the Thom class), by Corollary 13.3.1, we get

$$\mathrm{ch}^*_V(i_!\mathcal{F}) = \rho^*(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) \smile \Psi_V.$$

By (13.2.1), we get the first formula. The second follows from the first.

Remark 13.3.2. 1. The equalities in Corollary 13.3.2 hold in cohomology with $\mathbb{Q}$ coefficients (cf. Remarks 13.1.2).

2. In the case V is nonsingular, the formulas are proved in [12].

3. In [86], a similar formula is proved for the Thom class of a vector bundle. Namely, let $p : E \to X$ be a complex vector bundle of rank ℓ over a topological space X. Then, in our notation,

$$\mathrm{ch}_X^*(\lambda_{E^*}) = p^*\mathrm{td}^{-1}(E) \smile \Psi_E,$$

where $\lambda_{E^*} = \sum_{i=0}^r (-1)^i \Lambda^i p^* E^*$ and Ψ_E denotes the Thom class of E. If X is a C^∞ manifold, this formula is proved at the level of Čech-de Rham cocycles as above; in the situation of Theorem 13.3.1, simply let $M = E$, $V = X$ (identified with the zero section of E), $N = p^*E$ and $s = s_\Delta$ and note that $\Psi_E = c^\ell(p^*E, s_\Delta)$.

4. In the algebraic category, the formulas are proved for a locally free $\mathcal{O}_V$-module on an LCI by analyzing the graph construction in [15, 3. Proposition]. Note that their general Riemann-Roch theorem does not directly imply the formulas.

5. These formulas are also proved at the level of differential forms and currents in [80].

13.4 Homology Chern Characters and Classes

Let V be a subvariety of pure codimension k in a complex manifold M. Suppose that V is an LCI. Thus the ideal sheaf $\mathcal{I}_V$ of functions vanishing on V is locally generated by k functions and the normal sheaf $\mathcal{N}_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$ is locally free. We denote by N_V the associated vector bundle and let $\tau_V = TM|_V - N_V$ be the virtual tangent bundle of V (cf. Sect. 5.1). Note that it does not depend on the embedding $i : V \hookrightarrow M$.

Definition 13.4.1. For a coherent $\mathcal{O}_V$-module $\mathcal{F}$, we define the homology Chern character $\mathrm{ch}_*(\mathcal{F})$ by

$$\mathrm{ch}_*(\mathcal{F}) = \mathrm{td}N_V \frown A(\mathrm{ch}_V^*(i_!\mathcal{F})).$$

Remark 13.4.1. 1. If V is an LCI defined by a section of a vector bundle N over M, we may write

$$\mathrm{ch}_*(\mathcal{F}) = A(\mathrm{td}N \smile \mathrm{ch}_V^*(i_!\mathcal{F})).$$

2. The above definition is related to the (homology) Todd class $\tau(\mathcal{F})$ of $\mathcal{F}$ in [15] by

$$\mathrm{ch}_*(\mathcal{F}) = (\mathrm{td}^{-1}\tau_V) \frown \tau(\mathcal{F}).$$

In [15], $\tau(\mathcal{F})$ is defined using an embedding of V, but it is shown that $\tau(\mathcal{F})$ is independent of the embedding for a projective variety V. Thus $\mathrm{ch}_*(\mathcal{F})$ is also independent of the embedding in this case.

The following directly follows from the definition.

Proposition 13.4.1. (1) *For an exact sequence of coherent $\mathcal{O}_V$-modules*

$$0 \longrightarrow \mathcal{F}_q \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow 0,$$

we have

$$\sum_{i=0}^{q}(-1)^i \mathrm{ch}_*(\mathcal{F}_i) = 0.$$

(2) *For a vector bundle E over V and a coherent $\mathcal{O}_V$-module $\mathcal{F}$,*

$$\mathrm{ch}_*(E \otimes \mathcal{F}) = \mathrm{ch}^*(E) \frown \mathrm{ch}_*(\mathcal{F}).$$

The following is a direct consequence of Corollary 13.3.2.

Proposition 13.4.2. *Suppose either V is nonsingular or V is defined by a section and $\mathcal{F}$ is locally free. Then we have*

$$\mathrm{ch}_*(\mathcal{F}) = \mathrm{ch}^*(\mathcal{F}) \frown [V].$$

In particular, for the structure sheaf $\mathcal{O}_V$,

$$\mathrm{ch}_*(\mathcal{O}_V) = [V].$$

If $\mathrm{ch}_*(\mathcal{F})$ is in the image of the Poincaré homomorphism $H^*(V) \to H_*(V)$, we may define the homology Chern class $c_*(\mathcal{F})$ via Newton's formula. Namely, suppose

$$\mathrm{ch}_*(\mathcal{F}) = \sigma^* \frown [V],$$

for some σ^* in $H^*(V)$ and write $\sigma^* = \sum_{i\geq 0} \frac{\sigma^i}{i!}$ with σ^i in $H^{2i}(V)$. Then we define $\gamma^* = 1 + \sum_{i\geq 1} \gamma^i$ with γ^i in $H^{2i}(V)$ by

$$\sigma^i - \gamma^1\sigma^{i-1} + \gamma^2\sigma^{i-2} - \cdots + (-1)^i i\gamma^i = 0, \qquad i \geq 1.$$

If we define the homology Chern class $c_*(\mathcal{F})$ of $\mathcal{F}$ by

$$c_*(\mathcal{F}) = \gamma^* \frown [V],$$

then it is not difficult to check that the definition does not depend on the choice of σ^*.

Example 13.4.1. Suppose either V is nonsingular or V is defined by a section and $\mathcal{F}$ is locally free. Then, from Proposition 13.4.2,

$$c_*(\mathcal{F}) = c^*(\mathcal{F}) \frown [V].$$

In particular,

$$c_*(\mathcal{O}_V) = [V].$$

13.5 Characteristic Classes of the Tangent Sheaf

Let V be an LCI defined by a section of a vector bundle N over a complex manifold M. Denoting by Ω_M and Ω_V the sheaves of holomorphic 1-forms on M and V, respectively, we have the exact sequence

$$0 \longrightarrow \mathcal{I}_V/\mathcal{I}_V^2 \longrightarrow \Omega_M \otimes_{\mathcal{O}_M} \mathcal{O}_V \longrightarrow \Omega_V \longrightarrow 0.$$

Let $\Theta_M = \mathcal{O}_M(TM)$ be the tangent sheaf of M. We define the tangent sheaf Θ_V of V by $\Theta_V = \mathcal{H}om_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, which is independent of the embedding $V \hookrightarrow M$. From the above sequence, we have the exact sequence

$$0 \longrightarrow \Theta_V \longrightarrow \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_V \longrightarrow \mathcal{N}_V \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V) \longrightarrow 0.$$

Setting $\mathcal{E} = \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, we get, from Propositions 13.4.1 and 13.4.2,

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \mathrm{ch}_*(\mathcal{E}).$$

If p is an isolated singular point of V, by the Riemann-Roch theorem for the embedding $p \hookrightarrow M$, we have $\mathrm{ch}_*(\mathcal{E}) = \tau(V, p)[p]$, where $\tau(V, p) = \dim \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)_p$ is the Tjurina number of V at p. Thus we have the following:

Theorem 13.5.1. *Let V be an LCI of dimension n (≥ 1) defined by a section with isolated singularities $p_1, \dots, p_s$. For the tangent sheaf Θ_V of V, we have*

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \sum_{i=1}^{s} \tau(V, p_i)\,[p_i],$$

$$c_*(\Theta_V) = c^*(\tau_V) \frown [V] + (-1)^{n+1}(n-1)! \sum_{i=1}^{s} \tau(V, p_i)\,[p_i].$$

Recall that the class $c^*(\tau_V) \frown [V]$ coincides with the Fulton–Johnson class $c_*^{\mathrm{FJ}}(V)$ of V, in this case.

Let (V, p) be an isolated complete intersection singularity. If it admits a good $\mathbb{C}^*$-action in the sense of [116, 9.B], then $\tau(V, p) = \mu(V, p)$, the Milnor number of V at p ([74, 3. Satz], [116, (9.10) Proposition]). On the other hand, for a variety as in Theorem 13.5.1, the Schwartz–MacPherson class $c_*(V)$ of V is given by a formula in [155] (cf. Theorem 12.2.1). Hence we have

Corollary 13.5.1. *Let V be as in Theorem 13.5.1 with $n = 1$ or 2. If V admits a good $\mathbb{C}^*$-action near each singular point p_i, then*

$$c_*(\Theta_V) = c_*(V).$$

Remark 13.5.1. It would be an interesting problem to compare the class $\mathrm{ch}_*(\mathcal{F})$ with the homology Chern character of $\mathcal{F}$ as defined in [140] (see also [100]).

References

1. M. Abate, *The residual index and the dynamics of holomorphic maps tangent to the identity*, Duke Math. J. **107** (2001) 173–207.
2. M. Abate, F. Bracci and F. Tovena, *Index theorems for holomorphic self-maps*, Ann. of Math. **159** (2004) 819–864.
3. M. Abate, F. Bracci and F. Tovena, *Index theorems for holomorphic maps and foliations*, math.CV/0601602
4. R. Abraham and J. Robin, *Transversal mappings and flows*, W.A. Benjamin, Inc., 1967.
5. N. A'Campo, *Structures de pseudovariété sur les espaces analytiques complexes*, in Intersection Homology, A. Borel et al. Progress in Mathematics, Vol. 50, Birkhaüser, Boston, Inc. 1984, 41–45.
6. M. Aguilar, J. Seade and A. Verjovsky, *Indices of vector fields and topological invariants of real analytic singularities*, Crelle's, J. Reine u. Ange. Math. **504** (1998), 159–176.
7. Aleksandrov, A.G. The index of vector fields and logarithmic differential forms. Funct. Anal. Appl. 39, No. 4, 245–255 (2005).
8. P. Aluffi, *Chern classes for singular hypersurfaces*, Trans. Amer. Math. Soc. **351** (1999), 3989–4026.
9. V. I. Arnold, *Index of a singular point of a vector field, the Petrovskii-Oleinik inequality and Mixed Hodge structures*, Funct. Anal. Appl. **12** (1978), 1–14.
10. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps, Volume I*, Birkhäuser, 1985 (Translated from Russian).
11. M. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology, **1** (1961), 25–45.
12. M. Atiyah and F. Hirzebruch, *The Riemann-Roch for analytic embeddings*, Topology **1** (1962), 151–166.
13. P. Baum and R. Bott, *On the zeroes of meromorphic vector-fields*, Essays on Topology and Related Topics, Springer-Verlag, New York, Heidelberg, Berlin, 29–47, 1970.
14. P. Baum and R. Bott, *Singularities of holomorphic foliations* J. Differential Geom. **7** (1972), 279–342.
15. P. Baum, W. Fulton and R. MacPherson, *Riemann-Roch for singular varieties*, Publ. Math. IHES **45** (1975), 101–145.
16. Ch. Bonatti and X. Gómez-Mont, *The index of a holomorphic vector field on a singular variety*, Astérisque 222 (1994), 9–35.
17. H.-C. Graf von Bothmer, W. Ebeling and X. Gómez-Mont, *An algebraic formula for the index of a vector field on an isolated complete intersection singularity*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 5, 1761–1783.
18. R. Bott, *The stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. **13** (1957), 933–935.

19. R. Bott, *Lectures on characteristic classes and foliations*, in "Lectures on Algebraic and Differential Topology", Lecture Notes in Mathematics **279** Springer-Verlag 1972, 1–94.
20. R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics 82, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
21. F. Bracci, *The dynamics of holomorphic maps near curves of fixed points*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **2** (2003), 493–520.
22. F. Bracci and T. Suwa, *Residues for singular pairs and dynamics of biholomorphic maps of singular surfaces*, Intern. J. Math. **15** (2004), 443–466.
23. F. Bracci and T. Suwa, *Residues for holomorphic foliations of singular pairs*, Advances in Geom. **5** (2005), 81–95.
24. F. Bracci and F. Tovena, *Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces*, Math. Z. **242** (2002), 481–490.
25. J.-P. Brasselet, *Définition combinatoire des homomorphismes de Poincaré, Alexander et Thom pour une pseudo-variété* in "Caractéristique d'Euler-Poincaré", Astérisque 82-83, Société Mathématique de France (1981), 71–91.
26. J.-P. Brasselet, *Existence des classes de Chern en théorie bivariante*, Astérisque 101-102 (1983), 7–22.
27. J.-P. Brasselet, *Local Euler obstruction, old and new*, in "Proceedings of XI Brazilian Topology Meeting (Rio Claro, 1998)", World Sci. Publishing, River Edge, NJ (2000), 140–147.
28. J.-P. Brasselet, *Characteristic Classes of Singular Varieties*, book in preparation.
29. J.-P. Brasselet, D. T. Lê and J. Seade, *Euler obstruction and indices of vector fields*, Topology 39 (2000), 1193–1208.
30. J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor numbers and classes of local complete intersections*, Proc. Japan Acad. Ser. A Math. Sci 75 (1999), 179–183.
31. J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor classes of local complete intersections*, Transactions A. M. S. **354** (2001), 1351–1371.
32. J.-P. Brasselet, D. Massey, A. J. Parameswaran and J. Seade, *Euler obstruction and defects of functions on singular varieties*, J. London Math. Soc. 70 (2004), 59–76.
33. J.-P. Brasselet et M.-H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Astérisque **82-83** (1981), 93–147.
34. J.-P. Brasselet, J. Seade and T. Suwa, *An explicit cycle representing the Fulton-Johnson class*, in "Singularités Franco-Japonaises", Sémin. Congr., 10, Soc. Math. France, Paris, p. 21–38, 2005.
35. J.-P. Brasselet, J. Seade and T. Suwa, *A proof of the proportionality theorem*, Preprint 2005.
36. J.-P. Brasselet, J. Seade and T. Suwa, *Proportionality of Indices of 1-Forms on Singular Varieties*, in "Singularities in geometry and topology 2004", ed. J.-P. Brasselet et al., Proceedings of the 3rd Franco-Japanese colloquium on singularities, Hokkaido, Japan. Advanced Studies in Pure Mathematics 46, 49–65 (2007).
37. J. Briançon, P. Maisonobe et M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom*, Invent. Math. **117** (1994), 531–550.
38. J. Bruce and R. Roberts, *Critical points of functions on analytic varieties*, Topology **27** (1988), 57–90.
39. M. Brunella, *Feuilletages holomorphes sur les surfaces complexes compactes*, Ann. Sci. E.N.S **30** (1997), 569–594.
40. M. Brunella, *Some remarks on indices of holomorphic vector fields*, Publicacions Matematiques **41** (1997), 527–544.
41. R.-O. Buchweitz and G.-M. Greuel, *The Milnor number and deformations of complex curve singularities*, Invent. Math. **58** (1980), 241–281.

42. C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. **115** (1982), 579–595.
43. V. Castellanos, *The index of non-algebraically isolated singularities*, Bol. Soc. Mat. Mex. **8** (2002), 141–148.
44. S.S. Chern, *Characteristic classes of Hermitian Manifolds*, Ann. of Math. **47** (1946), 85–121.
45. A. Dimca, *Singularities and Topology of Hypersurfaces*, Springer-Verlag, 1992.
46. A. Dubson, *Classes caractéristiques des variétés singuliéres*, C.R. Acad.Sc. Paris **287** (1978), no. 4, 237–240.
47. A. Dubson, *Formule pour l'indice des complexes constructibles et des D-modules holonomes*, C. R. Acad. Sci. Paris Sér. I Math. 298 (1984), no. 6, 113–116.
48. A. du Plessis, *Continuous controlled vector fields,* Singularity theory (Liverpool, 1996), London Math. Soc. Lecture Note Ser., 263, Cambridge Univ. Press, Cambridge, 1999, 189–197.
49. W. Ebeling and S. Gusein-Zade, *On the index of a vector field at an isolated singularity*, The Arnoldfest (Toronto, ON, 1997), 141–152, Fields Inst. Commun., 24, Amer. Math. Soc., Providence, RI, 1999.
50. W. Ebeling and S. Gusein-Zade, *On the index of a holomorphic* 1*-form on an isolated complete intersection singularity*, (Russian) Dokl. Akad. Nauk 380 (2001), no. 4, 458–461.
51. W. Ebeling and S. Gusein-Zade, *Indices of 1-forms on an isolated complete intersection singularity*, Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 439–455, 742–743.
52. W. Ebeling and S. Gusein-Zade, *Radial index and Euler obstruction of a 1-form on a singular variety*, Geom. Dedicata 113 (2005), 231–241.
53. W. Ebeling and S. Gusein-Zade, *Indices of vector fields or 1-forms and characteristic numbers*. Bull. London. Math. Soc. 37 (2005), 747–754.
54. W. Ebeling and S. Gusein-Zade, *Indices of vector fields and 1-forms on singular varieties*, In Global aspects of complex geometry, P. 129–169. Springer-Verlag 2006, Ed. F. Cattanese, H. Esnault and al.
55. W. Ebeling and S. Gusein-Zade, *Indices of collections of 1-forms*. Singularities in geometry and topology, 629–639, World Sci. Publ., Hackensack, NJ, 2007. Ed. J.-P. Brasselet et al.,
56. W. Ebeling and S. Gusein-Zade, *Chern obstructions for collections of 1-forms on singular varieties*. In "Singularity theory"; Proceedings of the 2005 Marseille singularity school and conference, CIRM, Marseille, France. Ed. D. Chéniot et al., World Scientific. 557–564 (2007).
57. W. Ebeling, S. Gusein-Zade and J. Seade, *Homological index for 1-forms and a Milnor number for isolated singularities*, International J. Math. 15 (2004), 895–905.
58. D. Eisenbud and H. Levine, *An algebraic formula for the degree of a* C^∞ *map germ*, Ann. Math. (2) 106 (1977), no. 1, 19–44.
59. W. Fulton, *Intersection Theory*, Springer-Verlag (1984).
60. W. Fulton and K. Johnson, *Canonical classes on singular varieties*, Manuscripta Math. **32** (1980), 381–389.
61. W. Fulton and R. MacPherson, *Categorical Framework for the study of singular Spaces*, Memoirs of Amer. Math. Soc. **243** (1981).
62. L. Giraldo, X. Gómez-Mont and P. Mardešić, *Computation of topological numbers via linear algebra: hypersurfaces, vector fields and vector fields on hypersurfaces*, Complex geometry of groups (Olmu, 1998), 175–182, Contemp. Math., 240, Amer. Math. Soc., Providence, RI, 1999.
63. L. Giraldo, X. Gómez-Mont and P. Mardešić, *On the complex formed by contracting differential forms with a vector field on a hypersurface singularity*, Bol. Soc. Mat. Mexicana (3) 7 (2001), no. 2, 211–221.

64. L. Giraldo, X. Gómez-Mont and P. Mardešić, *A law of conservation of number for local Euler characteristics*, Complex manifolds and hyperbolic geometry (Guanajuato, 2001), 251–259, Contemp. Math., 311, Amer. Math. Soc., Providence, RI, 2002.
65. L. Giraldo, X. Gómez-Mont and P. Mardešić, *On the index of vector fields tangent to hypersurfaces with non-isolated singularities*, J. London Math. Soc. (2) 65 (2002).
66. L. Giraldo, X. Gómez-Mont and P. Mardešić, *Flags in zero dimensional complete intersections and indices of real vector fields*, Preprint 2008, arXiv:math/0612275.
67. B. Gmira, *Sur les feuilletages holomorphes singuliers de codimension 1,* Publicacions Matemàtiques, 1992, 36 (1): 229–240,
68. X. Gómez-Mont, *An algebraic formula for the index of a vector field on a hypersurface with an isolated singularity,* J. Algebraic Geom. **7** (1998), 731–752.
69. X. Gómez-Mont and P. Mardešić, *The index of a vector field tangent to a hypersurface and signature of the relative Jacobian determinant*, Ann. Inst. Fourier (Grenoble) **47** (1997), 1523–1539.
70. X. Gómez-Mont and P. Mardešić, *The index of a vector field tangent to an odd-dimensional hypersurface, and the signature of the relative Hessian*, Funct. Anal. Appl. 33 (1999), no. 1, 1–10.
71. X. Gómez-Mont, J. Seade and A. Verjovsky, *The index of a holomorphic flow with an isolated singularity*, Math. Ann. **291** (1991), 737–751.
72. G. Gonzalez-Sprinberg, *L'obstruction locale d'Euler et le théorème de MacPherson*, Astérisque **82-83** (1981), 7–32.
73. M. Goresky and R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 14. Berlin Springer-Verlag 1988.
74. G.-M. Greuel, *Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten*, Math. Ann. **214** (1975), 235–266.
75. Ph. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1978.
76. N. de Góes Grulha Júnior, *L'obstruction d'Euler locale d'une application*, Ann. Fac. Sci. Toulouse Math. (6) 17 (2008), no. 1, 53–71.
77. N. de Góes Grulha Júnior, *The Euler obstruction and Bruce Roberts' Milnor Number*, The Quarterly Journal of Mathematics, 2009, 60(3), 291–302.
78. S. Gusein-Zade, *The index of a singular point of a gradient vector field*, Funct. Anal. and Appl. **18** (1984), 6–10.
79. H. Hamm, *Lokale topologische Eigenschaften komplexer Räume*, Math. Ann. **191** (1971), 235–252.
80. R. Harvey and H.B. Lawson *A theory of characteristic currents associated with a singular connection*, Astérisque **213**, Société Mathématique de France, 1993.
81. J.-P. Henry, M. Merle et C. Sabbah, *Sur la condition de Thom stricte pour un morphisme analytique complexe* , Ann. Sci. E. N. S. **17** (1984), 227–268.
82. H. Hironaka, *Stratification and flatness*, in "Real and Complex singularities", Proceedings of the Nordic Summer School, Oslo 1976; edited by P. Holm; Publ. Sijthoff & Noordhoff Int. Publishers (1977), 199–265.
83. M. Hirsch, *Smooth regular neighborhoods*, Ann. of Math. **76** (1962), 524–530.
84. F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.
85. H. Hopf, *Uber die Curvatura integra geschlossener Hyperflachen*, Math. Ann. *95* (1925), 340–367.
86. B. Iversen, *Local Chern classes*, Ann. Scient. Éc. Norm. Sup. **9** (1976), 155–169.
87. T. Izawa and T. Suwa, *Multiplicity of functions on singular varieties*, Internat. J. Math. **14**, 5 (2003), 541–558.
88. M. Kato, *Singularities and some global topological properties*, Proc. R.I.M.S. Singularities Symposium, Kyoto University 1978.

89. M. Kervaire, *Courbure intégrale généralisée et homotopie*, Math. Ann. **131** (1956), 219–252.
90. M. Kervaire, *An interpretation of G. Whitehead's generalization of H. Hopf's invariant.* Ann. Math. **69** (1959), 345–365.
91. M. Kervaire and J. Milnor, *Groups of homotopy spheres I*, Ann. of Maths. 77 (1963), 504–537.
92. B. Khanedani, *On homological index*, Ann. Fac. Sci. Toulouse, Math, **9** (2000), 433–450.
93. B. Khanedani and T. Suwa, *First variation of holomorphic forms and some applications*, Hokkaido Math. J. **26** (1997), 323–335.
94. G.N. Khimshiashvili, *On the local degree of a smooth mapping*, Comm. Acad. Sci. Georgian SSR 85, No. 2 (1977), 309–312 (in Russian).
95. A.G. Khovanskii, *Index of a polynomial field*, Funct. Anal. Appl. 13, 38–45 (1979).
96. H. King and D. Trotman, *Poincaré–Hopf theorems on stratified sets*, preprint 1996.
97. O. Klehn, *On the index of a vector field tangent to a hypersurface with non-isolated zero in the embedding space*, Math. Nachr. 260 (2003), 48–57.
98. O. Klehn, *Real and complex indices of vector fields on complete intersection curves with isolated singularity.* Compos. Math. 141 (2005), no. 2, 525–540.
99. K. Kodaira, *On compact complex analytic surfaces, I*, Ann. of Math. **71** (1960), 111–152.
100. M. Kwieciński, *Sur le transformé de Nash et la construction du graphe de MacPherson*, Thèse, Univ. de Provence, 1994.
101. D.T. Lê, *Computation of the Milnor number of an isolated singularity of a complete intersection*, Funct. Anal. Appl. **8** (1974), 45–49.
102. D. T. Lê, *Some remarks on the relative monodromy*, Real and Complex Singularities Oslo 1976, Sijhoff en Nordhoff, Alphen a.d. Rijn 1977, pp. 397–403.
103. D. T. Lê, *Le concept de singularité isolée de fonction analytique*, Adv. Stud. Pure Math. **8** (1986), 215–227, North Holland.
104. D. T. Lê, *Singularités isolées des intersections complètes*, in "Introduction à la théorie des singularités", Travaux en cours **36**, Hermann (1988).
105. D. T. Lê, *Polyèdres évanescents et effondrements*, in "A fete of Topology", edited by Y. Matsumoto, T. Mizutani, S. Morita; Academic Press, 1988, 293–329.
106. D. T. Lê et B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulieres*, Ann. of Math **114** (1981), 457–491.
107. D. T. Lê et B. Teissier, *Cycles évanescents et conditions de Whitney*, in Proc. Symp. Pure Math **40**, Part 2 (1983), 65–103.
108. D. Lehmann, *Variétés stratifiées C^∞ : Intégration de Cech-de Rham et théorie de Chern–Weil*, in "Geometry and Topology of Submanifolds II", Proc. Conf., May 30-June 3, 1988, Avignon, France, Publ. World Scientific (1990), 205–248.
109. D. Lehmann, *Systèmes d'alvéoles et intégration sur le complexe de Čech-de Rham*, Publications de l'IRMA, 23, N^o VI, Université de Lille I, 1991.
110. D. Lehmann, *Résidus des sous-variétés invariantes d'un feuilletage singulier*, Ann. Inst. Fourier **41** (1991), 211–258.
111. D. Lehmann, M. Soares and T. Suwa, *On the index of a holomorphic vector field tangent to a singular variety*, Bol. Soc. Bras. Mat. **26** (1995), pp. 183–199.
112. D. Lehmann and T. Suwa, *Residues of holomorphic vector fields relative to singular invariant subvarieties*, J. Differential Geom. **42** (1995), 165–192.
113. D. Lehmann and T. Suwa, *Generalization of variations and Baum-Bott residues for holomorphic foliations on singular varieties*, Intern. J. Math. **10** (1999), 367–384.
114. A. Lins Neto, *Algebraic solutions of polynomial differential equations and foliations in dimension two*, Springer Lecture Notes 1345 (1986), Conference on holomorphic dynamics, Mexico, 192–232.

115. A. Lins Neto, *Complex codimension one foliations leaving a compact submanifold invariant,* Dynamical systems and bifurcation theory (Rio de Janeiro 1985), 295–317, Pitman Res. Notes Math. Ser. 160, Longman Sci. Tech. Harlow (1987).
116. E.J.N. Looijenga, *Isolated Singular Points on Complete Intersections*, LMS Lecture Notes **77**, Cambridge Univ. Press 1984.
117. R. MacPherson, *Chern classes for singular varieties*, Ann. of Math **100** (1974), 423–432.
118. D.B. Massey, *Lê Cycles and Hypersurface Singularities*, Springer-Verlag, Lecture Notes in Mathematics **1615**, 1995.
119. D.B. Massey, *Hypercohomology of Milnor fibers*, Topology 35 (1996), no. 4, 969–1003.
120. J. Milnor, *Topology from the Differentiable Viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.
121. J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton 1968.
122. J. Milnor and P. Orlik, *Isolated singularities defined by weighted homogeneous polynomials*, Topology **9** (1970), 385–393.
123. J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies **76**, Princeton University Press, Princeton, 1974.
124. M. Morse, *Singular points of vector fields under general boundary conditions*, Amer. Journ. Math. **51** (1929), 165–178.
125. T. Ohmoto, T. Suwa and S. Yokura, *A remark on the Chern classes of local complete intersections*, Proc. Japan Acad. **73** (1997), 93–95.
126. T. Ohmoto and S. Yokura, *Product formulas for the Milnor class*, Bull. Polish Acad. Sci. Math. 48 (2000), no. 4, 387–401.
127. A. Parusiński, *A generalization of the Milnor number*, Math. Ann. **281** (1988), 247–254.
128. A. Parusiński, *Limits of tangent spaces to fibers and the w_f condition*, Duke Math. Jounal **72** (1993), 99–108.
129. A. Parusiński and P. Pragacz, *Characteristic numbers of degeneracy loci*, Contemp. Math. **123** (1991), 189–198.
130. A. Parusiński and P. Pragacz, *A formula for the Euler characteristic of singular hypersurfaces*, J. Algebraic Geom. **4** (1995), 337–351.
131. A. Parusiński and P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*, J. Algebraic Geom. **10** (2001), 63–79.
132. V. Poenarú, *Sur la theorie des immersions*, Topology **1** (1962), 81–100.
133. Ch. Pugh, *Generalized Poincaré index formulas*, Topology **7** (1968), 217–226.
134. C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, Astérisque **130** (1985), 161–192.
135. C. Sabbah, *Espaces conormaux bivariants*, Thèse, Université de Paris VII, 1986.
136. K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA **27** (1980), 265–291.
137. J. Schürmann, *A short proof of a formula of Brasselet, Lê and Seade*, math. AG/ 0201316.
138. J. Schürmann, *Topology of singular spaces and constructible sheaves*, Mathematical Monographs-new series, Birkhäuser 2003.
139. M.-H. Schwartz, *Classes caractéristiques définies par une stratification d'une variété analytique complexe*, C.R. Acad. Sci. Paris **260** (1965), 3262–3264, 3535–3537.
140. M.-H. Schwartz, *Classes et caractères de Chern–Mather des espaces linéaires*, C.R. Acad. Sci. Paris **295** 1982, 399–402.
141. M.-H. Schwartz, *Champs radiaux sur une stratification analytique complexe*, Travaux en cours **39**, Hermann, Paris, 1991.
142. M.-H. Schwartz, *Classes de Chern des ensembles analytiques*, Actualités Mathématiques, Hermann, Paris, 2000.

143. J. Seade, *Singular points of complex surfaces and homotopy*, Topology **21** (1982), 1–8.
144. J. Seade, *The index of a vector field on a complex surface with singularities*, in "The Lefschetz Centennial Conf.", ed. A. Verjovsky, Contemp. Math. **58**, Part III, Amer. Math. Soc. (1987), 225–232.
145. J. Seade, *The index of a vector field under blow-ups*, Bol. Soc. Mat. Mexicana **37** (1992), 449–462.
146. J. Seade, *Remarks on contact structures and vector fields on isolated complete intersection singularities.* Accepted for publication in the Proceedings of the IX Workshop on Real and Complex Singularities, to appear in Contemporary Mathematics.
147. J. Seade and T. Suwa, *A residue formula for the index of a holomorphic flow*, Math. Annalen **304** (1996), 621–634.
148. J. Seade and T. Suwa, *Residues and topological invariants of singular holomorphic foliations*, Internat. J. Math. **8** (1997), 825–847.
149. J. Seade and T. Suwa, *An adjunction formula for local complete intersections*, Internat. J. Math. **9** (1998), 759–768.
150. J. Seade, M. Tibăr and A. Verjovsky, *Global Euler obstruction and polar invariants*, Math. Ann. 333 (2005), no. 2, 393–403.
151. J. Seade, M. Tibăr and A. Verjovsky, *Milnor numbers and Euler obstruction*, Bull. Braz. Math. Soc., New Series, **36** (2005) 275–283.
152. C. T. Simpson, *Higgs bundles and local systems.* Inst. Hautes Études Sci. Publ. Math. No. 75 (1992), 5–95.
153. N. Steenrod, *The Topology of Fiber Bundles*, Princeton Univ. Press, 1951.
154. T. Suwa, *Indices of holomorphic vector fields relative to invariant curves on surfaces*, Proc. of the Amer. Math. Soc. **123** (1995), 2989–2997.
155. T. Suwa, *Classes de Chern des intersections complètes locales*, C.R. Acad. Sci. Paris **324** (1996), 67–70.
156. T. Suwa, *Indices of vector fields and residues of singular holomorphic foliations*, Actualités Mathématiques, Hermann, Paris, 1998.
157. T. Suwa, *Dual class of a subvariety,* Tokyo J. Math. **23** (2000), 51–68.
158. T. Suwa, *Characteristic classes of coherent sheaves on singular varieties.* Singularities-Sapporo 1998, 279–297, Adv. Stud. Pure Math., 29, Kinokuniya, Tokyo, 2000.
159. T. Suwa, *Residues of Chern classes*, J. Math. Soc. Japan **55** (2003), 269–287.
160. T. Suwa, *Residues of Chern classes on singular varieties*, Singularités Franco-Japonaises, Marseille 2002, Séminaires et Congrès **10**, Soc. Math. France, 265–285, 2005.
161. T. Suwa, *Residue Theoretical Approach to Intersection Theory*, Proceedings of the IX Workshop on Real and Complex Singularities, São Carlos, Brazil, 207–261, Contemp. Math., 459, Amer. Math. Soc., Providence, RI, 2008.
162. B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney.* Singularités à Cargése. Astérisque, Nos. 7 et 8, Soc. Math. France, Paris, p. 285–362, 1973.
163. B. Teissier. *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney.* In *Algebraic geometry (La Rábida, 1981)*, volume 961 of Lecture Notes in Math., pages 314–491. Springer, Berlin, 1982.
164. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comm. Math. Helv. **28** (1954), 17–86.
165. A.K. Tsikh, *Weakly holomorphic functions on complete intersections, and their holomorphic extension*, Math. USSR Sbornik, **61** (1988), 421–436.
166. A. N. Varchenko, *On the local residue and the intersection form on the vanishing cohomology*, Math. USSR Izvestiya **26** (1986), 31–52.
167. J.L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard.* Invent. Math., 36:295–312, 1976.

168. J.L. Verdier, *Spécialisation des classes de Chern*, Astérisque 82–83, 149–159, 1981.
169. S. Yokura *On a Milnor class*, Preprint 1997.
170. C.T.C. Wall, *Topological invariance of the Milnor number mod 2*, Topology **22** (1983), 345–350.
171. H. Whitney, *Local properties of Analytic Varieties*, Symposium in honor of M. Morse, Princeton Univ. Press, edited by S. Cairns, 1965.
172. V.M. Zakalyukin, *Algebraic calculation of the index of a singular point of a vector field*, Funct. Anal. Appl. 6, 69–70 (1972).

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