The Cauchy-Schwarz Inequality

Proofs and applications in various spaces

Cauchy-Schwarz olikhet
Bevis och tillämpningar i olika rum

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THE CAUCHY-SCHWARZ INEQUALITY

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Abstract. We give some background information about the Cauchy-Schwarz inequality including its history. We then continue by providing a number of proofs for the inequality in its classical form using various proof techniques, including proofs without words. Next we build up the theory of inner product spaces from metric and normed spaces and show applications of the Cauchy-Schwarz inequality in each content, including the triangle inequality, Minkowski's inequality and Hölder's inequality. In the final part we present a few problems with solutions, some proved by the author and some by others.

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Key words and phrases. Cauchy-Schwarz inequality, mathematical induction, triangle inequality, Pythagorean theorem, arithmetic-geometric means inequality, inner product space.

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2. Introduction

The Cauchy-Schwarz inequality may be regarded as one of the most important inequalities in mathematics. It has many names in the literature: Cauchy-Schwarz, Schwarz, and Cauchy-Bunyakovsky-Schwarz inequality. The reason for this inconsistency is mainly because it developed over time and by many people. This inequality has not only many names, but also it has many manifestations. In fact, the inequalities below are all based on the same inequality.

(1) $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.$
(2) $\left|\int_0^1 f(x)\,\overline{g(x)}dx\right|^2 \leq \int_0^1 |f(x)|^2 dx \int_0^1 |g(x)|^2 dx.$
(3) $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|.$

This versatility of its forms makes it a well-used tool in mathematics and it has many applications in a wide variety of fields, e.g., classical and modern analysis including partial differential equations and multivariable calculus as well as geometry, linear algebra, and probability theory. Outside of mathematics it finds its usefulness in, for example, physics, where it plays a role in the uncertainty relations of Schrödinger and Heisenberg [1].

Recently, there have been several extensions and generalizations of the C-S inequality and its reverse in various settings such as matrices, operators, and C^*-algebras, see e.g., [6, 14]. These topics are of special interest in operator theory and matrix analysis; cf. [1, 4]. Understanding the classical results on C-S inequality is definitely a starting point for doing some research in these research areas.

This thesis is organized as follows. In chapter 3, we start up with a brief history of the C-S inequality. Chapter 4 is devoted to classical proofs of the inequality, both in real and complex case, with some additional proofs that don't use any words. In chapter 5, we look into some spaces and show different applications of the inequality. Finally, in chapter 6, we provide a few examples of problems with various difficulties.

3. Historical perspectives

The Cauchy-Schwarz (C-S) inequality made its first appearance in the work *Cours d'analyse de l'École Royal Polytechnique* by the French mathematician Augustin-Louis Cauchy (1789-1857). In this work, which was published in 1821, he introduced the inequality in the form of finite sums, although it was only written as a note.
In 1859 a Russian former student of Cauchy, Viktor Yakovlevich Bunyakovsky (1804-1889), published a work on inequalities in the journal *Mémoires de l'Académie Impériale des Sciences de St-Pétersbourg*. Here he proved the inequality for infinite sums, written as integrals, for the first time.

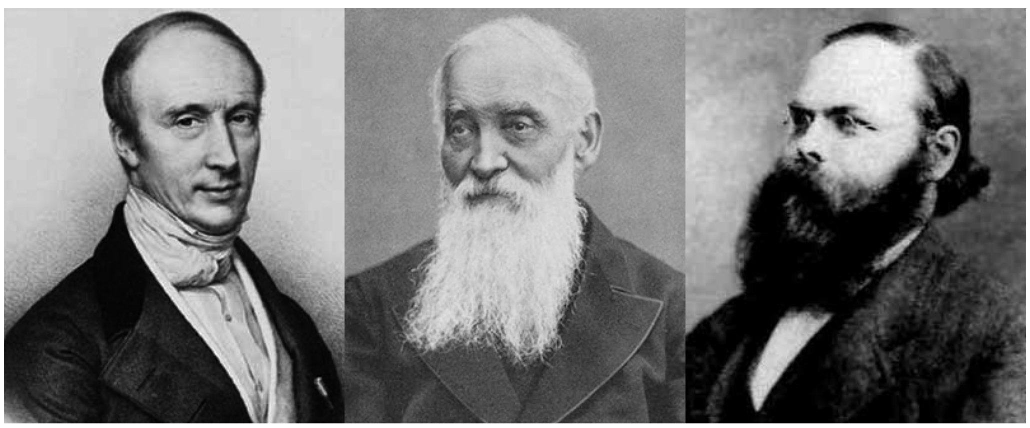

Figure 1. (From left) Augustin-Louis Cauchy, Viktor Yakovlevich Bunyakovsky and Karl Hermann Amandus Schwarz.

In 1888, Karl Hermann Amandus Schwarz (1843-1921) published a work on minimal surfaces named *Über ein die flächen kleinsten flächeninhalts betreffendes problem der variationsrechnung* in which he found himself in need of the integral form of Cauchy's inequality, but since he was unaware of the work of Bunyakovsky, he presented the proof as his own. The proofs of Bunyakovsky and Schwarz are not similar and Schwarz's proof is therefore considered independent, although of a later date. A big difference in the methods of Bunyakovsky and Schwarz was in the rigidity of the limiting process, which was of bigger importance for Schwarz. The arguments of Schwarz can also be used in more general settings like in the framework of inner product spaces [19].

4. Some proofs of the C-S inequality

There are many ways to prove the C-S inequality. We will begin by looking at a few proofs, both for real and complex cases, which demonstrates the validity of this classical form. Most of the following proofs are from H.-H Wu and S. Wu [24]. We will also look at a few proofs without words for the inequality in the plane. Later on, when we've established a more modern view of the inequality, additional proofs will be given.

4.1. **C-S inequality for real numbers.**

Theorem 4.1. *If* $a_1, \dots, a_n$ *and* $b_1, \dots, b_n$ *are real numbers, then*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \tag{4.1}$$

or, equivalently,

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}. \tag{4.2}$$

First proof [24]. We will use mathematical induction as a method for the proof. First we observe that

$$(a_1 b_2 - a_2 b_1)^2 \geq 0.$$

By expanding the square we get

$$(a_1 b_2)^2 + (a_2 b_1)^2 - 2a_1 b_2 a_2 b_1 \geq 0.$$

After rearranging it further and completing the square on the left-hand side, we get

$$a_1^2 b_1^2 + 2a_1 b_1 a_2 b_2 + a_2^2 b_2^2 \leq a_1^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_2^2$$

or, equivalently,

$$(a_1 b_1 + a_2 b_2)^2 \leq \left(a_1^2 + a_2^2\right)\left(b_1^2 + b_2^2\right).$$

By taking the square roots of both sides, we reach

$$|a_1 b_1 + a_2 b_2| \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}, \tag{4.3}$$

which proves the inequality (4.2) for $n = 2$.
Assume that inequality (4.2) is true for any n terms. For $n+1$, we have that

$$\sqrt{\sum_{i=1}^{n+1} a_i^2}\sqrt{\sum_{i=1}^{n+1} b_i^2} = \sqrt{\sum_{i=1}^{n} a_i^2 + a_{n+1}^2}\sqrt{\sum_{i=1}^{n} b_i^2 + b_{n+1}^2}. \tag{4.4}$$

By comparing the right-hand side of equation (4.4) with the right-hand side of inequality (4.3) we know that

$$\sqrt{\sum_{i=1}^{n} a_i^2 + a_{n+1}^2}\sqrt{\sum_{i=1}^{n} b_i^2 + b_{n+1}^2} \geq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + \left|a_{n+1}b_{n+1}\right|.$$

Since we assume that inequality (4.2) is true for n terms, we have that

$$\begin{aligned}\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + |a_{n+1}b_{n+1}| &\geq \sum_{i=1}^{n} a_i b_i + |a_{n+1}b_{n+1}| \\ &\geq \sum_{i=1}^{n+1} a_i b_i,\end{aligned}$$

which proves the C-S inequality. □

Second proof [24]. We will use proof by induction again, but this time we deal with sequences.
Let

$$S_n = \left(\sum_{i=1}^{n} a_i b_i\right)^2 - \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.$$

We have

$S_{n+1} - S_n =$

$$= \left(\sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1}\right)^2 - \left(\sum_{i=1}^{n} a_i^2 + a_{n+1}^2\right)\left(\sum_{i=1}^{n} b_i^2 + b_{n+1}^2\right)$$

$$= \left(\sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1}\right)^2 - \left(\sum_{i=1}^{n} a_i^2 + a_{n+1}^2\right)\left(\sum_{i=1}^{n} b_i^2 + b_{n+1}^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 + \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

$$= -b_{n+1}^2 \sum_{i=1}^{n} a_i^2 - a_{n+1}^2 \sum_{i=1}^{n} b_i^2 + 2a_{n+1} b_{n+1} \sum_{i=1}^{n} a_i b_i$$

$$= -\left(b_{n+1}^2 \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) + a_{n+1}^2 \left(b_1^2 + b_2^2 + \cdots + b_n^2\right) - 2a_{n+1} b_{n+1} \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)\right)$$

$$= -\left((b_{n+1} a_1 - a_{n+1} b_1)^2 + (b_{n+1} a_2 - a_{n+1} b_2)^2 + \cdots + (b_{n+1} a_n - a_{n+1} b_n)^2\right)$$

$$= -\sum_{i=1}^{n} (b_{n+1} a_i - a_{n+1} b_i)^2,$$

from which we get $S_{n+1} \leq S_n$. Hence $S_n \leq S_{n-1} \leq \ldots \leq S_1 = 0$, which yields the C-S inequality. $\square$

Third proof [24]. We start this proof by observing that the inequality

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2 \geq 0 \tag{4.5}$$

is always true.

Expanding the square and separating the sums give us

$$\begin{aligned} 0 &\leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i b_i a_j b_j\right) \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^2 b_j^2 + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} a_j^2 b_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_i a_j b_j \\ &= \frac{1}{2} \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \frac{1}{2} \sum_{j=1}^{n} a_j^2 \sum_{i=1}^{n} b_i^2 - \sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} a_j b_j. \end{aligned}$$

Renaming the indices and reshaping the formula give us

$$\frac{1}{2} \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 + \frac{1}{2} \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} a_i b_i \geq 0,$$

from where

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \geq 0,$$

which yields the C-S inequality. □

Fourth proof [24]. For the quadratic equation

$$ax^2 + bx + c = 0 \tag{4.6}$$

the solutions can be found using the well-known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Inside the square root the expression $b^2 - 4ac$ determines when there are real solutions to the quadratic equation. This expression is called the discriminant of the quadratic equation and is denoted by $\Delta = b^2 - 4ac$. If this discriminant is positive there are two real roots, when it is negative there are no real roots, and when it is zero there is a double root.

Now, let

$$f(x) = \sum_{i=1}^{n} (a_i x - b_i)^2.$$

Since $f(x) \geq 0$ for all $x \in \mathbb{R}$, and by the fact that the function is a quadratic polynomial, the equation (4.6) cannot have two real solutions. Therefore, the discriminant of $f(x)$ is non-positive.

When we expand the square we get

$$f(x) = \sum_{i=1}^{n} (a_i x - b_i)^2 = x^2 \sum_{i=1}^{n} a_i^2 - 2x \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2.$$

By comparing this with the quadratic equation in (4.6) we can do the following substitutions:

$$a = \sum_{i=1}^{n} a_i^2, \; b = -2 \sum_{i=1}^{n} a_i b_i, \text{ and } c = \sum_{i=1}^{n} b_i^2.$$

As a result, the discriminant can be written as

$$\Delta = \left(-2 \sum_{i=1}^{n} a_i b_i \right)^2 - 4 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 = 4 \left(\sum_{i=1}^{n} a_i b_i \right)^2 - 4 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.$$

Since $\Delta \leq 0$,

$$4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq 0,$$

which yields the C-S inequality. □

Fifth proof [24]. Let $A = \sum_{i=1}^{n} a_i^2$ and $B = \sum_{i=1}^{n} b_i^2$.

When $\sum_{i=1}^{n} a_i^2 = 0$ or $\sum_{i=1}^{n} b_i^2 = 0$ we have that $\sum_{i=1}^{n} a_i b_i = 0$, we can assume that $A \neq 0$ and $B \neq 0$.

Let

$$x_i = \frac{a_i}{\sqrt{A}} \quad \text{and } y_i = \frac{b_i}{\sqrt{B}}.$$

As a result,

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = 1.$$

Now, by observing that the inequality

$$0 \leq \sum_{i=1}^{n} (x_i - y_i)^2$$

is always valid, we can expand the square and rearrange the inequality as

$$0 \leq \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - 2\left|\sum_{i=1}^{n} x_i y_i\right|$$

or

$$2\left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2.$$

Since

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = 1,$$

we can write the inequality as

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq 1.$$

By change of variables we get that

$$\left|\sum_{i=1}^{n} \frac{a_i b_i}{\sqrt{A}\sqrt{B}}\right| \leq 1,$$

which can be arranged to

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{A}\sqrt{B}$$

or

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2},$$

which proves the C-S inequality. □

Sixth proof. This proof is from T. Andreescu and B. Enescu [3]. Let a and $b \in \mathbb{R}$, $x > 0$, and $y > 0$. After observing that the inequality $(ay - bx)^2 \geq 0$ is always true, we expand and rearrange it as

$$\frac{a^2 y}{x} + \frac{b^2 x}{y} \geq 2ab.$$

After completing the square and rearranging it further, we get that

$$\frac{a^2 y}{x} + \frac{b^2 x}{y} + a^2 + b^2 \geq (a+b)^2.$$

Hence

$$\frac{(a+b)^2}{(x+y)} \leq \frac{a^2}{x} + \frac{b^2}{y}. \tag{4.7}$$

We replace b by $b + c$ and y by $y + z$ in inequality (4.7) to get

$$\frac{(a+b+c)^2}{(x+y+z)} \leq \frac{a^2}{x} + \frac{(b+c)^2}{y+z} \leq \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}.$$

Using this method n times, and by using a suitable notation, we get

$$\frac{(a_1 + a_2 + \ldots + a_n)^2}{(x_1 + x_2 + \ldots + x_n)} \leq \frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \ldots + \frac{a_n^2}{x_n}.$$

Now if we set $a_i = \alpha_i \beta_i$ and $x_i = \beta_i^2$ we get that

$$\left(\sum_{i=1}^{n} \alpha_i \beta_i\right)^2 \leq \sum_{i=1}^{n} \alpha_i^2 \sum_{i=1}^{n} \beta_i^2,$$

which yields the C-S inequality. □

Seventh proof [24]. Consider the arithmetic-geometric means inequality

$$\sum_{i=1}^{n} \sqrt{x_i y_i} \leq \sum_{i=1}^{n} \frac{x_i + y_i}{2}. \tag{4.8}$$

Let $A = \sqrt{\sum\limits_{i=1}^{n} a_i^2}$ and $B = \sqrt{\sum\limits_{i=1}^{n} b_i^2}$.

By choosing $x_i = \frac{a_i^2}{A^2}$ and $y_i = \frac{b_i^2}{B^2}$ in inequality (4.8) we get

$$\sum_{i=1}^{n} \frac{a_i b_i}{AB} \leq \frac{1}{2} \sum_{i=1}^{n} \frac{a_i^2}{A^2} + \frac{b_i^2}{B^2}. \tag{4.9}$$

Now, by observing that the right-hand side of inequality (4.9) is equal to 1, we have

$$\sum_{i=1}^{n} a_i b_i \leq AB = \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2},$$

which proves the C-S inequality. □

Eighth proof [24]. Consider the arithmetic-geometric means inequality (4.8) in the last proof. Let $A = \sum\limits_{i=1}^{n} a_i^2$, $B = \sum\limits_{i=1}^{n} b_i^2$, and $C = \sum\limits_{i=1}^{n} a_i b_i$.

By choosing $x_i = \frac{a_i^2 B}{C^2}$ and $y_i = \frac{b_i^2}{B}$ in inequality (4.8), we get

$$\sum_{i=1}^{n} \sqrt{\frac{a_i^2 B}{C^2} \frac{b_i^2}{B}} \leq \frac{1}{2} \sum_{i=1}^{n} \left(\frac{a_i^2 B}{C^2} + \frac{b_i^2}{B} \right),$$

from which we reach

$$\sum_{i=1}^{n} \sqrt{\frac{a_i b_i}{C}} \leq \frac{1}{2} \sum_{i=1}^{n} \frac{a_i^2 B}{C^2} + \frac{b_i^2}{B}. \tag{4.10}$$

The left-hand side in inequality (4.10) is 1 and we therefore get

$$2 \leq \sum_{i=1}^{n} \frac{a_i^2 B}{C^2} + \sum_{i=1}^{n} \frac{b_i^2}{B}. \tag{4.11}$$

The second term on right-hand side in inequality (4.11) is 1, therefore

$$2 \leq \sum_{i=1}^{n} \frac{a_i^2 B}{C^2} + 1,$$

whence

$$1 \leq \frac{AB}{C^2},$$

which gives rise to the C-S inequality. □

Ninth proof [24]. Consider the arithmetic-geometric means inequality (4.8). Let $A = \sqrt{\sum_{i=1}^{n} a_i^2}$ and $B = \sqrt{\sum_{i=1}^{n} b_i^2}$.
By choosing $x_i = \frac{Ba_i^2}{A}$ and $y_i = \frac{Ab_i^2}{B}$ in inequality (4.8) we get

$$\begin{aligned}\sum_{i=1}^{n} |a_i b_i| &\leq \frac{1}{2}\sum_{i=1}^{n}\left(\frac{Ba_i^2}{A} + \frac{Ab_i^2}{B}\right)\\ &\leq \frac{1}{2}\left(\frac{B}{A}\sum_{i=1}^{n} a_i^2 + \frac{A}{B}\sum_{i=1}^{n} b_i^2\right)\\ &\leq \frac{1}{2}\left(\frac{B}{A}A^2 + \frac{A}{B}B^2\right)\\ &= AB,\end{aligned}$$

from which we reach the C-S inequality. □

Lemma 4.2 (Rearrangement inequality)**.** *The rearrangement inequality states that, given the real numbers* $x_1 \leq \ldots \leq x_n$ *and* $y_1 \leq \ldots \leq y_n$*, the similarly sorted pairing* $x_1y_1 + \ldots + x_ny_n$ *is the largest possible pairing and the oppositely sorted pairing* $x_1y_n + \ldots + x_ny_1$ *is the smallest. As a result of this we have that*

$$x_ny_1 + \ldots + x_1y_n \leq x_1y_1 + \ldots + x_ny_n. \tag{4.12}$$

Proof of rearrangement inequality. Consider $n = 2$ and assume $x_2 \geq x_1$ and $y_2 \geq y_1$. From these considerations we get that $(x_2 - x_1)(y_2 - y_1) \geq 0$, which can be expanded and reorganized into $x_1y_1 + x_2y_2 \geq x_1y_2 + x_2y_1$. This gives us the desired result.
In the general case we assume, as before, that $x_1 \leq \ldots \leq x_n$ and $y_1 \leq \ldots \leq y_n$. Suppose that the pairing that maximizes the sum is not the similarly sorted one. For that to happen we must have at least one instance where we pair x_i with y_j and x_k with y_l where $i < j$ and $k > l$. But using the result from the $n = 2$ case we know that

$$x_iy_j + x_ky_l \leq x_iy_l + x_ky_j,$$

which contradicts that the proposed pairing is the greatest one. □

Tenth proof [24]. Let

$$A = B = C = \{a_1b_1,\ \ldots,\ a_1b_n,\ a_2b_1,\ \ldots,\ a_2b_n,\ \ldots,\ a_nb_1,\ \ldots,\ a_nb_n\}$$

and

$$D = \{a_1b_1,\ \dots,\ a_nb_1,\ a_1b_2,\ \dots,\ a_nb_2,\ \dots,\ a_1b_n,\ \dots,\ a_nb_n\}.$$

Now, since A and B are similarily sorted, and C and D are mixed sorted we can apply the inequality (4.12)

$$C \cdot D \leq A \cdot B,$$

or equivalently

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_jb_ib_j \leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2b_j^2,$$

from where

$$\sum_{i=1}^{n} a_ib_i \sum_{j=1}^{n} a_jb_j \leq \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2.$$

Thus

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2,$$

from which we deduce the C-S inequality. □

Corollary 4.3. *Equality holds for inequality (4.1) if and only if the sequences are linearly dependent, i.e. there is a constant $\lambda \in \mathbb{R}$ such that $a_i = \lambda b_i$ for each $i \leq n$.*

Proof. This proof is from S. S. Dragomir [5]. The inequality (4.5) we used in the third proof tells us that equality holds if and only if

$$a_ib_j - a_jb_i = 0$$

for all $i,\ j \in \{1,\ \dots,\ n\}$.

Assuming that all b_j's are all non-zero, we have

$$\frac{a_i}{b_i} = \frac{a_j}{b_j} = \lambda,$$

which proves that equality in the C-S inequality holds if and only if the sequences are linearly dependent. If one or more b_j are zero, inequality (4.5) tells us that, either a_j is zero as well, or that all b_i are zero. Both cases give rise to linearly dependent sequences. □

4.2. **C-S inequality for complex numbers.**

Theorem 4.4. *If* $a_1, \dots, a_n$ *and* $b_1, \dots, b_n$ *are complex numbers, then*

$$\left|\sum_{i=1}^{n} a_i\overline{b_i}\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2,$$

where $\overline{z}$ *denotes the conjugate of* $z \in \mathbb{C}$.

Proof. This proof is borrowed from W. Rudin [18]. Let $A = \sum_{i=1}^{n} |a_i|^2$, $B = \sum_{i=1}^{n} |b_i|^2$, and $C = \sum_{i=1}^{n} a_i\overline{b_i}$. The numbers A and B are real, but C is complex. We may assume $B > 0$ since if $B = 0$, then the inequality would be trivial. Now we observe that the inequality

$$0 \leq \sum_{i=1}^{n} |Ba_i - Cb_i|^2$$

is always true. By expanding the square in its complex conjugates we get

$$0 \leq \sum_{i=1}^{n} (Ba_i - Cb_i)\left(B\overline{a_i} - \overline{Cb_i}\right).$$

After multiplying the parentheses and replacing the sums by our predefined variables, we reach

$$\begin{aligned}
0 &\leq B^2 \sum_{i=1}^{n} |a_i|^2 - B\overline{C} \sum_{i=1}^{n} a_i\overline{b_i} - BC \sum_{i=1}^{n} \overline{a_i}b_i + |C|^2 \sum_{i=1}^{n} |b_i|^2 \\
&\leq B^2 A - B\overline{C}C - BC\overline{C} + |C|^2 B \\
&\leq B^2 A - B|C|^2 \\
&\leq B\left(BA - |C|^2\right).
\end{aligned}$$

Now since $B > 0$ we have

$$0 \leq BA - |C|^2,$$

which yields the C-S inequality. □

4.3. **Proofs without words.** We will here provide some proofs of the C-S inequality in $\mathbb{R}^2$. Even though the proofs are self-evident we will give some explanation.

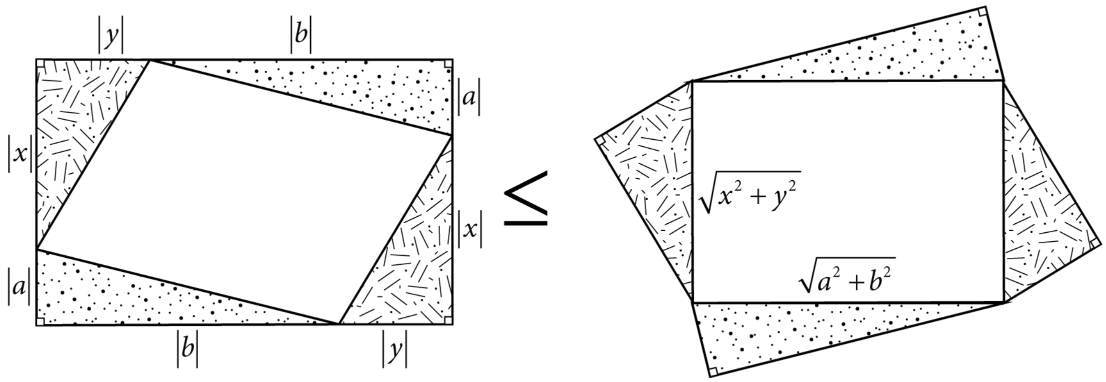

FIGURE 2. Proof by Roger Nelsen [15].

In the left-hand side of figure 2 we can see a parallelogram formed in the middle which, on the right-hand side is straightened up. The total area in the right-hand figure is clearly larger than or equal to the left-hand side and we can express the inequality of the total area as

$$(|x| + |a|)\,(|y| + |b|) \leq 2\left(\frac{1}{2}\,|x|\,|y| + \frac{1}{2}\,|a|\,|b|\right) + \sqrt{x^2+y^2}\sqrt{a^2+b^2}.$$

We can also easily verify that

$$|xb + ya| \leq |x|\,|b| + |y|\,|a|\,. \tag{4.13}$$

These two expressions together allow us to conclude that

$$|xb + ya| \leq |x|\,|b| + |y|\,|a| \leq \sqrt{x^2+y^2}\sqrt{a^2+b^2},$$

which proves the C-S inequality.

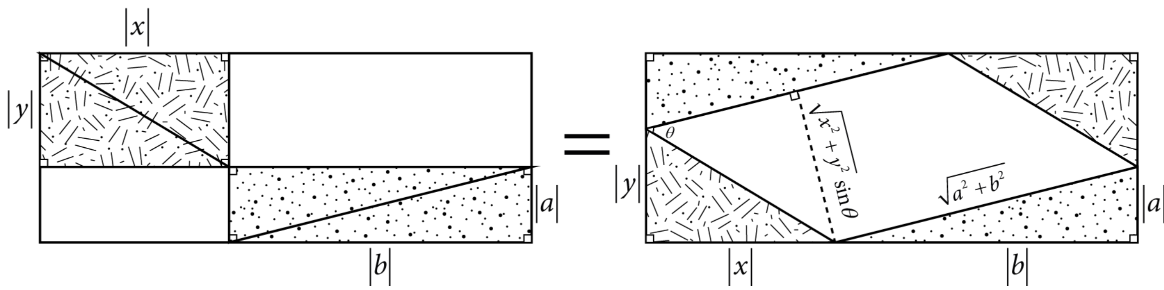

FIGURE 3. Proof by Sidney King [11].

In figure 3 we can easily see that the two non-shaded areas are of the same size:

$$|x|\,|a| + |y|\,|b| = \sqrt{x^2 + y^2}\sqrt{a^2 + b^2}\sin\theta.$$

Together with inequality (4.13) this allows us to conclude that

$$|xa + yb| \leq |x|\,|a| + |y|\,|b| \leq \sqrt{x^2 + y^2}\sqrt{a^2 + b^2},$$

which yields the C-S inequality.

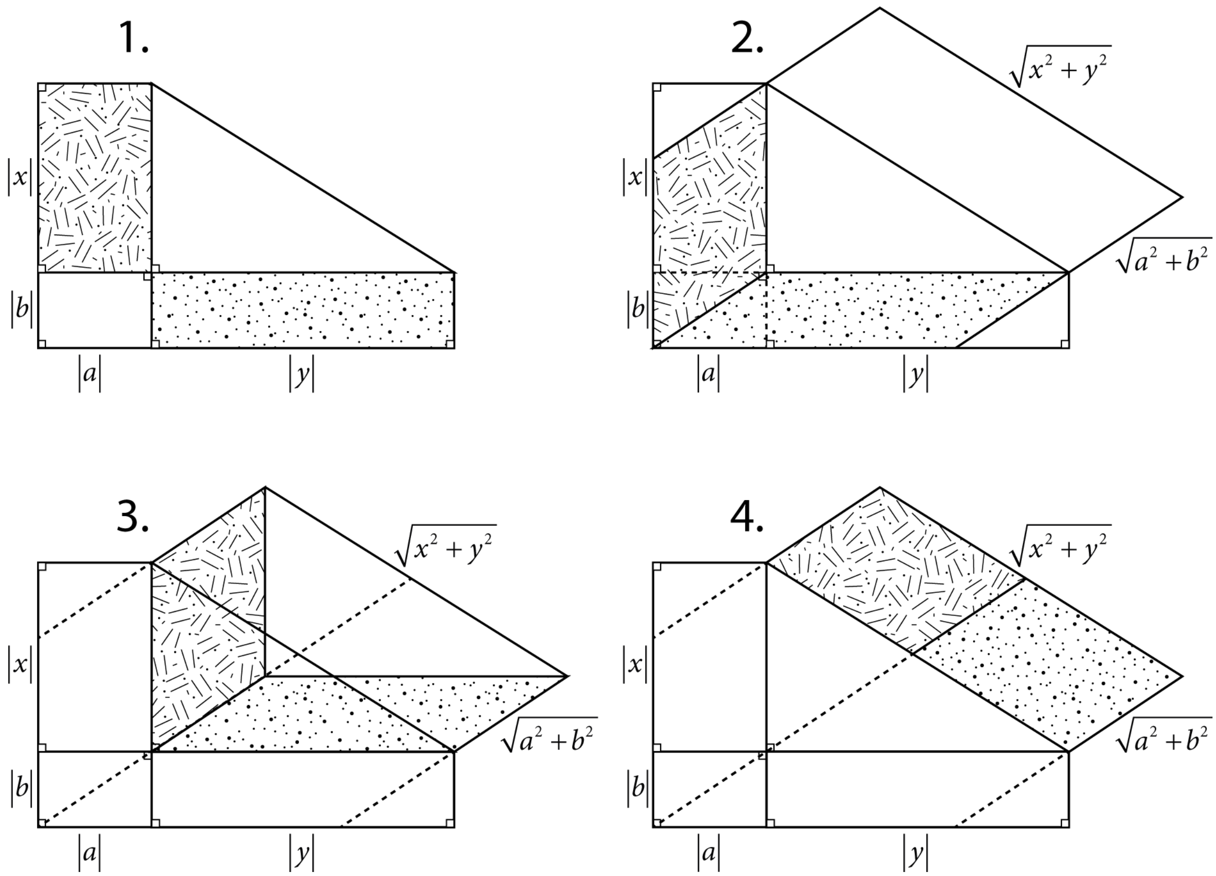

FIGURE 4. Proof by Claudi Alsina [2]. Projection of rectangles.

In figure 4 we are projecting the rectangles onto a common diagonal and hence creates a parallelogram which, as can be seen in figure 5, has the following property:

$$|xa + yb| \leq |x|\,|a| + |y|\,|b| \leq \sqrt{x^2 + y^2}\sqrt{a^2 + b^2},$$

from which we reach the C-S inequality.

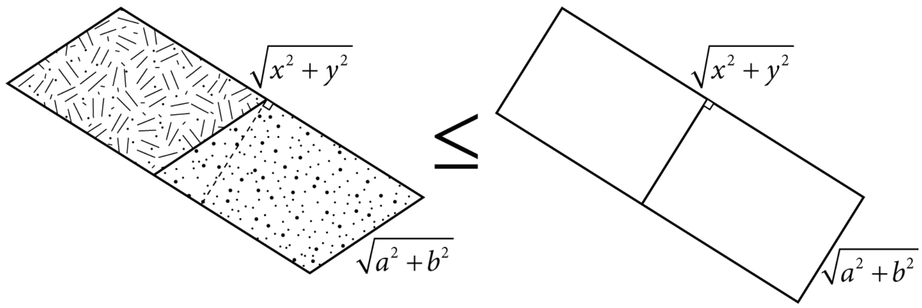

Figure 5. Proof by Claudi Alsina [2]. Conclusion.

5. C-S inequality in various spaces

We will now show the importance of the C-S inequality by investigating its roles in different spaces of decreasing generality, starting with metric spaces.

Definition 5.1. Let X be a set with an associated notion of distance, a non-negative, real function $d(x,\ y)$ defined for each x and $y \in X$ that has the following three properties:

1. Identity of indiscernibles: $d(x,\ y) = 0$ if and only if $x = y$.
2. Symmetry: $d(x,\ y) = d(y,\ x)$.
3. Triangle inequality: $d(x,\ z) \leq d(x,\ y) + d(y,\ z)$.

The set X is often referred to as a space and its members as points. The distance function is also known as a metric function or just a metric. The pair, the set and its distance function, is called a metric space and is denoted by $(X,\ d)$, or when context is clear, by just X.

Example 5.2. Let X be the set of all real numbers and by associating this with a metric, defined by

$$d(x,\ y) = |x - y|,$$

we get the metric space of the real line $\mathbb{R}$.

Example 5.3. Let $C([a,\ b])$ be the set of all continuous functions $f\ :\ [a,\ b] \to \mathbb{R}$. By associating this set with a metric, defined by

$$d(f,\ g) = \max_{t\in[a,b]} |f(t) - g(t)|,$$

we get a metric space. This space is also an example of a function space.

Definition 5.4. Let F be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and let V^n be the set of all ordered n-tuples $(x_1,\ x_2,\ \ldots,\ x_n)$ with $x_i \in F$ for $1 \leq i \leq n$.
A vector space is defined as $(V^n,\ +,\ \cdot)$, with the two associated operators called vector addition and scalar multiplication.

1. $+ : V^n \times V^n \to V^n := (x_1,\ \ldots,\ x_n) + (y_1,\ \ldots,\ y_n) = (x_1 + y_1,\ \ldots,\ x_n + y_n)$.
2. $\cdot : F \times V^n \to V^n := \alpha \cdot (x_1,\ \ldots,\ x_n) = (\alpha \cdot x_1,\ \ldots,\ \alpha \cdot x_n)$.

Definition 5.5. The space $\mathbb{R}^n$ associated with

$$d(x,\ y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}, \tag{5.1}$$

is called the Euclidean n-space.

Example 5.6. The Euclidean n-space is a metric space.

Proof. This proof is from A. N. Kolmogorov and S. V Fumin [9]. The distance function (5.1) clearly has properties (1) and (2) of definition 5.1. What remains to show is that it also has property (3).

By completing the square on each side of the C-S inequality we get

$$\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2},$$

so that

$$\sum_{k=1}^{n} (a_k + b_k)^2 \leq \left(\sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}\right)^2.$$

Hence

$$\sqrt{\sum_{k=1}^{n} (a_k + b_k)^2} \leq \sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}. \tag{5.2}$$

Let $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_n)$, $\mathbf{y} = (y_1,\ y_2,\ \ldots,\ y_n)$, and $\mathbf{z} = (z_1,\ z_2,\ \ldots,\ z_n) \in \mathbb{R}^n$. If we choose $a_k = x_k - y_k$ and $b_k = y_k - z_k$, inequality (5.2) become

$$\sqrt{\sum_{k=1}^{n} (x_k - z_k)^2} \leq \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} + \sqrt{\sum_{k=1}^{n} (y_k - z_k)^2}$$

or equivalently,

$$d(x,\ z) \leq d(x,\ y) + d(y,\ z).$$

□

Definition 5.7. Let V be a vector space over the field $\mathbb{R}$ or $\mathbb{C}$ and let $\mathbf{x} \in V$. A norm, $\|\mathbf{x}\|$, on V is a function with the following properties

(1) $\|\mathbf{x}\| > 0$ if $\mathbf{x} \neq \mathbf{0}$.
(2) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$ for any scalar α.

(3) $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$.

A vector space V where such a norm exists is called a normed vector space, a normed linear space, or just a normed space.

Example 5.8 (Euclidean norm)**.** The Euclidean space $\mathbb{R}^n$ has a norm, defined by

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2},$$

where $\mathbf{x} = (x_1, \ \ldots, \ x_n) \in \mathbb{R}^n$. Even though this is a natural norm, it is not the only one.

Example 5.9. There are other norms on $\mathbb{R}^n$, for example, when $\mathbf{x} = (x_1, \ \ldots, \ x_n) \in \mathbb{R}^n$

(1) The 1-norm on $\mathbb{R}^n$: $\|\mathbf{x}\|_1 = \sum_{k=1}^{n} |x_k|$.
(2) The ∞-norm on $\mathbb{R}^n$: $\|\mathbf{x}\|_\infty = \max\left(|x_1|, \ \ldots, \ |x_n|\right)$.
(3) The p-norm on $\mathbb{R}^n$: $\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$.

Example 5.10 (Normed function space)**.** Let X be the set defined in example 5.3. By associating this space with a norm, defined by

$$\|f\| = \max_{t\in[a, \ b]} |f(t)|,$$

we get a normed function space denoted, as before, by $C([a, \ b])$.

Proposition 5.11. *Every normed space is a metric space.*

Proof. Let V be a normed vector space, and define a function on V by $d(\mathbf{x}-\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|$. This function satisfies naturally properties (1) and (2) in definition 5.1. To prove that the function also has property (3) and so is indeed a metric, we replace $\mathbf{a} = \mathbf{x}-\mathbf{y}$ and $\mathbf{b} = \mathbf{y}-\mathbf{z}$ in the triangle inequality for normed vector spaces,

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|,$$

and get

$$\|\mathbf{x}-\mathbf{z}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\|,$$

which is equivalent to

$$d(\mathbf{x}-\mathbf{z}) \leq d(\mathbf{x}-\mathbf{y})+d(\mathbf{y}-\mathbf{z}),$$

which proves property (3), and every normed space is therefore a metric space. □

Corollary 5.12. *Not every metric space is a normed space.*

Proof. This can be shown by a counterexample, as demonstrated by W. Rudin [18]. Let $x,\ y \in \mathbb{R}$ and define a distance function as

$$d(x,\ y)=\begin{cases}0 & \text{if } x=y \\ 1 & \text{if } x \neq y\end{cases}.$$

In this case, it is clear that property (2) in definition 5.7 fails to be true, since $\|\alpha \mathbf{x}\|=|\alpha|\,\|\mathbf{x}\|$ is not true for all scalars α. □

Lemma 5.13. *Any* $\mathbf{a}=(a_1,\ \ldots,\ a_n),\ \mathbf{b}=(b_1,\ \ldots,\ b_n) \in \mathbb{R}^n$ *satisfy Hölder's inequality*

$$\sum_{k=1}^{n}|a_k b_k| \leq\left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}}, \tag{5.3}$$

where $p,\ q \in (1,\ \infty)$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. This proof is classical and we used some ideas from Wikipedia [21], see also [7]. The weighted inequality of arithmetic and geometric means, or weighted AM-GM inequality for short, states that

$$\sqrt[w]{\prod_{k=1}^{n} x_k^{w_k}} \leq \frac{1}{w} \sum_{k=1}^{n} w_k x_k,$$

for all $\mathbf{x}=(x_1,\ \ldots,\ x_n) \in \mathbb{R}^+$, all weights $\mathbf{w}=(w_1,\ \ldots,\ w_n) \in \mathbb{R}^+$, and where $w=\sum\limits_{k=1}^{n} w_k$.

If we put $x_1=u^p$ and $x_2=v^q$, and the weights $w_1=\frac{1}{p}$ and $w_2=\frac{1}{q}$, where $p,\ q \in \mathbb{R}$ and $w=w_1+w_2=1$, the weighted AM-GM inequality turns into

$$uv \leq \frac{u^p}{p}+\frac{v^q}{q}.$$

This special case of the AM-GM inequality is named Young's inequality, [13].
If we substitute $u = x_k$ and $v = y_k$, and sum over all k, we get

$$\sum_{k=1}^{n} x_k y_k \leq \frac{1}{p}\sum_{k=1}^{n} x_k^p + \frac{1}{q}\sum_{k=1}^{n} y_k^q. \tag{5.4}$$

Let $A = \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}}$ and $B = \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}$. Set $x_k = \frac{a_k}{A}$ and $y_k = \frac{b_k}{B}$, and inequality (5.4) becomes

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k b_k}{AB} &\leq \frac{1}{p}\sum_{k=1}^{n}\left(\frac{a_k}{A}\right)^p + \frac{1}{q}\sum_{k=1}^{n}\left(\frac{b_k}{B}\right)^q \\
&\leq \frac{1}{pA^p}\sum_{k=1}^{n} a_k^p + \frac{1}{qB^q}\sum_{k=1}^{n} b_k^q \\
&\leq \frac{1}{pA^p}A^p + \frac{1}{qB^q}B^q \\
&= 1,
\end{aligned}$$

whence

$$\sum_{k=1}^{n} a_k b_k \leq AB = \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}},$$

which yields Hölder's inequality. □

Remark 5.14. Hölder's inequality is a generalization of the C-S inequality, as can be seen when $p = q = 2$ in inequality (5.3).

The examples in example 5.9 together with the norm in definition 5.7 form a special class of norms on $\mathbb{R}^n$ that is called p-norms, or ℓ_p-norms.

Theorem 5.15.

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$$

is a norm on $\mathbb{R}^n$.

Proof. This proof is from Wikipedia [21, 22], see also [13]. Properties (1) and (2) of definition 5.7 are trivial. What remains to show is that property (3), i.e. the triangle inequality, holds.

It is indeed Minkowski's inequality

$$\left(\sum_{k=1}^{n}|a_k+b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}|b_k|^p\right)^{\frac{1}{p}}.$$

Assume $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^+$. Let p and $q \in \mathbb{R}$, and $\frac{1}{p} + \frac{1}{q} = 1$.

First we observe that

$$\sum_{k=1}^{n}(a_k+b_k)^p = \sum_{k=1}^{n}a_k(a_k+b_k)^{p-1} + \sum_{k=1}^{n}b_k(a_k+b_k)^{p-1}.$$

By using Hölder's inequality on the two terms on the right-hand side we get

$$\begin{aligned}\sum_{k=1}^{n}(a_k+b_k)^p &\leq \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}(a_k+b_k)^{q(p-1)}\right)^{\frac{1}{q}} + \left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}(a_k+b_k)^{q(p-1)}\right)^{\frac{1}{q}}\\ &\leq \left(\left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}}\right)\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{\frac{1}{q}}.\end{aligned}$$

Hence

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}},$$

so that

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}},$$

which proves Minkowski's inequality. □

Definition 5.16. Let V be a vector space over the field $\mathbb{R}$ or $\mathbb{C}$. Associate each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V with a scalar $\langle \mathbf{u}, \mathbf{v} \rangle$, that is called an inner product, which has the following properties:

(1) Positive definiteness: $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, with equality if and only if $\mathbf{u} = \mathbf{0}$.
(2) Conjugate symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.
(3) Linearity (in the first argument): $\langle \alpha\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\alpha \in F$.

Then V is called an inner product space.

Remark 5.17. Conjugate linearity in the second variable follows from properties (2) and (3) of definition 5.16:

$$\langle \mathbf{u}, \alpha\mathbf{v} + \mathbf{w} \rangle = \overline{\langle \alpha\mathbf{v} + \mathbf{w}, \mathbf{u} \rangle} =$$

$$= \overline{\alpha \langle \mathbf{v}, \mathbf{u} \rangle} + \overline{\langle \mathbf{w}, \mathbf{u} \rangle} = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle ,$$

for all $\alpha \in F$.

Definition 5.18. Let V be an inner product space. The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example 5.19 (Dot product)**.** Let $\mathbf{a} = (a_1, \ \ldots, \ a_n)$ and $\mathbf{b} = (b_1, \ \ldots, \ b_n) \in \mathbb{R}^n$. Then

$$\langle \mathbf{a}, \mathbf{b} \rangle := \sum_{i=1}^{n} a_i b_i,$$

which we call the dot product, is an example of an inner product.

Proof. It follows trivially that the dot product has properties (1) and (2) in definition 5.16. We prove the linearity property. Let $\mathbf{u} = (u_1, \ \ldots, \ u_n)$, $\mathbf{v} = (v_1, \ \ldots, \ v_n)$, $\mathbf{w} = (w_1, \ \ldots, \ w_n) \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$.
We have

$$\langle \alpha\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} (\alpha u_i + v_i) w_i = \sum_{i=1}^{n} (\alpha u_i w_i + v_i w_i) =$$

$$= \sum_{i=1}^{n} \alpha u_i w_i + \sum_{i=1}^{n} v_i w_i = \alpha \sum_{i=1}^{n} u_i w_i + \sum_{i=1}^{n} v_i w_i = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle ,$$

and hence the dot product is an inner product. $\square$

Example 5.20. Let $C([0, \ 1])$ be the set of all complex-valued continuous functions on the interval $[0, \ 1]$. If $f, \ g \in C([0, \ 1])$, then

$$\langle f(x), g(x) \rangle := \int_0^1 f(x) \overline{g(x)} \ dx$$

is an inner product.

Proof. The first property in definition 5.16 is trivial. Since

$$\langle f, g \rangle = \int_0^1 f \cdot \bar{g} \ dx = \int_0^1 \bar{g} \cdot f \ dx = \int_0^1 \bar{g} \cdot \bar{\bar{f}} \ dx = \overline{\int_0^1 g \cdot \bar{f} \ dx} = \overline{\langle g, f \rangle},$$

we have shown that property (2) holds and since

$$\begin{aligned}\langle \alpha f + g, h\rangle &= \int_0^1 (\alpha f + g)\,\overline{h}\; dx \\ &= \int_0^1 \alpha f\overline{h} + g\overline{h}\; dx \\ &= \int_0^1 \alpha f\overline{h}\; dx + \int_0^1 g\overline{h}\; dx \\ &= \alpha \int_0^1 f\overline{h}\; dx + \int_0^1 g\overline{h}\; dx \\ &= \alpha \langle f, h\rangle + \langle g, h\rangle\,,\end{aligned}$$

we have also shown that property (3) holds and therefore it is an inner product.

□

Theorem 5.21 (Pythagorean theorem)**.** *Let V be an inner product space and* $\mathbf{u}$, $\mathbf{v} \in V$*. When* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal,*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Proof. Proof is from Wikipedia [23], see also [10]. Suppose $\mathbf{u}$ and $\mathbf{v} \in V$ and that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then we have that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \|\mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \|\mathbf{v}\|^2.$$

Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal we have that $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle = 0$ and we get

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2,$$

which gives us the Pythagorean theorem. □

Theorem 5.22 (C-S inequality)**.** *Let V be an inner product space and* $\mathbf{u}$ *and* $\mathbf{v} \in V$*. Then the following inequality holds*

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|\,.$$

Proof. Proof is borrowed from S. S. Dragomir [5]. For any $\mathbf{u}$ and $\mathbf{v} \in V$ and scalar t we have that

$$\|\mathbf{u}+t\mathbf{v}\|^2 \ge 0.$$

Expanding the square we get

$$\begin{aligned}0 &\le \langle \mathbf{u}+t\mathbf{v}, \mathbf{u}+t\mathbf{v}\rangle \\ &\le \langle \mathbf{u}, \mathbf{u}+t\mathbf{v}\rangle + \langle t\mathbf{v}, \mathbf{u}+t\mathbf{v}\rangle \\ &\le \overline{\langle \mathbf{u}+t\mathbf{v}, \mathbf{u}\rangle} + \overline{\langle \mathbf{u}+t\mathbf{v}, t\mathbf{v}\rangle} \\ &\le \overline{\langle \mathbf{u}, \mathbf{u}\rangle} + \overline{\langle t\mathbf{v}, \mathbf{u}\rangle} + \overline{\langle \mathbf{u}, t\mathbf{v}\rangle} + \overline{\langle t\mathbf{v}, t\mathbf{v}\rangle} \\ &\le \langle \mathbf{u}, \mathbf{u}\rangle + \bar{t}\langle \mathbf{u}, \mathbf{v}\rangle + t\overline{\langle \mathbf{u}, \mathbf{v}\rangle} + \bar{t}t\langle \mathbf{v}, \mathbf{v}\rangle .\end{aligned}$$

Let $t = -\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{v}\|^2}$. Then

$$\langle \mathbf{u}, \mathbf{v}\rangle = -t\|\mathbf{v}\|^2 \text{ and } \langle \mathbf{v}, \mathbf{u}\rangle = -\bar{t}\|\mathbf{v}\|^2.$$

By suitable substitutions we get that

$$\|\mathbf{u}\|^2 - |t|^2\|\mathbf{v}\|^2 - |t|^2\|\mathbf{v}\|^2 + |t|^2\|\mathbf{v}\|^2 \ge 0,$$

or equivalently

$$\|\mathbf{u}\|^2 - |t|^2\|\mathbf{v}\|^2 \ge 0.$$

By substituting t, we get that

$$\|\mathbf{u}\|^2 - \frac{|\langle \mathbf{u}, \mathbf{v}\rangle|^2}{\|\mathbf{v}\|^2} \ge 0,$$

which proves the C-S inequality. □

Alternative proof for real inner product spaces. This proof is provided from Wikipedia [20]. Let **u** and $\mathbf{v} \in V$ where V in an inner product space over $\mathbb{R}$ and assume that $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$. We now have that

$$0 \le \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle .$$

Since $\langle \mathbf{u}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle = 1$ we have that

$$\langle \mathbf{u}, \mathbf{v}\rangle \le 1 = \|\mathbf{u}\|\,\|\mathbf{v}\| ,$$

from which we obtain the C-S inequality. □

Example 5.23. In example 5.20 we showed that for $C([0,\ 1])$ we have the inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)\overline{g(x)}\,dx.$$

This allows us to write the C-S inequality as

$$\left|\int_0^1 f(x)\overline{g(x)}dx\right|^2 \leq \int_0^1 |f(x)|^2 dx \int_0^1 |g(x)|^2 dx.$$

Remark 5.24. Let $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_n)$ and $\mathbf{y} = (y_1,\ y_2,\ \ldots,\ y_n)$ be two non-zero n-tuples of real numbers. Then $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are both non-zero, and since the C-S inequality

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

can be expressed as

$$-1 \leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \leq 1,$$

we may define

$$\theta := \cos^{-1}\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}\right),$$

which we call the angle between $\mathbf{x}$ and $\mathbf{y}$.

6. Problems involving the C-S inequality

Example 6.1. Prove that $ab + bc + ca \leq a^2 + b^2 + c^2$.

Solution by author. By choosing the vectors $\mathbf{x} = (a,\ b,\ c)$ and $\mathbf{y} = (b,\ c,\ a)$, and using the C-S inequality we get

$$ab + bc + ca \leq \sqrt{a^2 + b^2 + c^2}\sqrt{b^2 + c^2 + a^2} = a^2 + b^2 + c^2.$$

□

Example 6.2 ([16])**.** Let a, b, and c be real positive numbers such that $abc = 1$. Prove that $a + b + c \leq a^2 + b^2 + c^2$ using the C-S inequality.

Solution by author. The C-S inequality can be written as

$$a \cdot 1 + b \cdot 1 + c \cdot 1 \leq \sqrt{a^2 + b^2 + c^2}\sqrt{1^2 + 1^2 + 1^2}$$

or

$$a + b + c \leq \sqrt{a^2 + b^2 + c^2}\sqrt{3}. \tag{6.1}$$

Since the AM-GM inequality can be written as

$$\sqrt[3]{a^2b^2c^2} \leq \frac{a^2 + b^2 + c^2}{3}$$

and the fact that $abc = 1$ gives us

$$\sqrt{3} \leq \sqrt{a^2 + b^2 + c^2}. \tag{6.2}$$

From inequalities (6.1) and (6.2) we get that

$$a + b + c \leq \sqrt{a^2 + b^2 + c^2}\sqrt{3} \leq \sqrt{a^2 + b^2 + c^2}\sqrt{a^2 + b^2 + c^2} = a^2 + b^2 + c^2.$$

□

Example 6.3 (IrMO, 1999 [8])**.** Let a, b, and c be real positive numbers such that $a + b + c + d = 1$. Prove that

$$\frac{1}{2} \leq \frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + d} + \frac{d^2}{d + a}.$$

Solution by author. By choosing the vectors

$$\mathbf{x} = \left(\sqrt{a+b},\ \sqrt{b+c},\ \sqrt{c+d},\ \sqrt{d+a}\right) \text{ and}$$

$$\mathbf{y} = \left(\frac{a}{\sqrt{a+b}},\ \frac{b}{\sqrt{b+c}},\ \frac{c}{\sqrt{c+d}},\ \frac{d}{\sqrt{d+a}}\right),$$

and by using the C-S inequality we get

$$(a+b+c+d)^2 \le ((a+b)+(b+c)+(c+d)+(d+a))\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right)$$

$$\frac{(a+b+c+d)^2}{2(a+b+c+d)} \le \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}.$$

Since $a+b+c+d=1$ we get the desired result. □

Example 6.4 (Belarus IMO TST, 1999 [16])**.** Let a, b, and c be real positive numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{3}{2} \le \frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}.$$

Solution by author. By choosing the vectors

$$\mathbf{x} = \left(\sqrt{1+ab},\ \sqrt{1+bc},\ \sqrt{1+ca}\right) \text{ and } \mathbf{y} = \left(\frac{1}{\sqrt{1+ab}},\ \frac{1}{\sqrt{1+bc}},\ \frac{1}{\sqrt{1+ca}}\right),$$

and by using the C-S inequality we get

$$9 \le ((1+ab)+(1+bc)+(1+ca))\left(\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\right),$$

whence

$$\frac{9}{3+ab+bc+ca} \le \frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}.$$

As a result from example 6.1 we know that

$$\frac{9}{3+a^2+b^2+c^2} \le \frac{9}{3+ab+bc+ca},$$

and since $a^2+b^2+c^2=3$ we obtain

$$\frac{3}{2} \le \frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}.$$

□

Example 6.5. Let a, b, and c be real positive numbers. Prove that

$$\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a} \le 1.$$

Solution by author. First we reorganize the inequality so that we get 1 on the left-hand side of the inequality

$$\frac{a}{2a+b}-\frac{1}{2}+\frac{b}{2b+c}-\frac{1}{2}+\frac{c}{2c+a}-\frac{1}{2}\leq 1-\frac{3}{2},$$

which can be rewritten as

$$\frac{2a-2a-b}{2\left(2a+b\right)}+\frac{2b-2b-c}{2\left(2b+c\right)}+\frac{2c-2c-a}{2\left(2c+a\right)}\leq -\frac{1}{2}$$

or

$$1\leq \frac{b}{2a+b}+\frac{c}{2b+c}+\frac{a}{2c+a}.$$

By choosing the vectors

$$\mathbf{x}=\left(\frac{b}{\sqrt{2ab+b^2}},\ \frac{c}{\sqrt{2bc+c^2}},\ \frac{a}{\sqrt{2ca+a^2}}\right)\ \text{and}$$
$$\mathbf{y}=\left(\sqrt{2ab+b^2},\ \sqrt{2bc+c^2},\ \sqrt{2ca+a^2}\right),$$

and by using the C-S inequality we get

$$\begin{aligned}(a+b+c)^2 &\leq \left(\frac{b^2}{2ab+b^2}+\frac{c^2}{2bc+c^2}+\frac{a^2}{2ca+a^2}\right)\left(2ab+b^2+2bc+c^2+2ca+a^2\right)\\ &\leq \left(\frac{b^2}{2ab+b^2}+\frac{c^2}{2bc+c^2}+\frac{a^2}{2ca+a^2}\right)(a+b+c)^2.\end{aligned}$$

Hence

$$\begin{aligned}1 &\leq \frac{b^2}{2ab+b^2}+\frac{c^2}{2bc+c^2}+\frac{a^2}{2ca+a^2}\\ &=\frac{b}{2a+b}+\frac{c}{2b+c}+\frac{a}{2c+a}.\end{aligned}$$

□

Example 6.6 (Optimization problem)**.** Let

$$\begin{cases} a^2+b^2+c^2+d^2+e^2=16\\ a+b+c+d+e=8\end{cases}$$

where $a,\ b,\ c,\ d$, and $e\in\mathbb{R}$. Calculate the maximum and minimum value of e.

Solution by author. By choosing the vectors

$$\mathbf{x}=(a,\ b,\ c,\ d)\ \text{ and }\ \mathbf{y}=(1,\ 1,\ 1,\ 1),$$

and using the C-S inequality we get

$$\begin{aligned}(a+b+c+d)^2 &\leq \left(a^2+b^2+c^2+d^2\right)\left(1^2+1^2+1^2+1^2\right)\\ &\leq 4\left(a^2+b^2+c^2+d^2\right).\end{aligned}$$

So that

$$(8-e)^2 \leq 4\left(16-e^2\right),$$

whence

$$e\,(5e-16) \leq 0.$$

An upper bound of e is therefore 16/5 and a lower bound is 0. To make sure the upper and lower bounds are also maximum and minimum we need to check when equality holds. Since C-S inequality is an equality when the vectors are linearly dependent, we have that $a = b = c = d = k$.

$$\begin{cases} 4k^2 + (16/5)^2 = 16 \\ 4k + 16/5 = 8 \end{cases}$$

By solving this system we find that $k = 6/5$, and hence $e = 16/5$ is a maximum. With similar argument we find that when $k = 2$ and $e = 0$ we have a minimum.

□

Example 6.7 (Circumference of an ellipse [17])**.** Let a be the semi-major axis of an ellipse and b the semi-minor axis. One way to determine the circumference of an ellipse is by using the elliptic integral

$$E\,(\varepsilon) = \int_0^{\pi/2} \sqrt{1-\varepsilon^2 \sin^2\theta}\; d\theta.$$

$E\,(\varepsilon)$ is here the arc length of a quarter of the circumference measured in units of a. The circumference is therefore given by $4aE\,(\varepsilon)$. ε is a measure of eccentricity of an ellipse, and given by the formula $\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$, $(0 \leq \varepsilon < 1)$. Find an upper bound of this elliptic integral using the C-S inequality.

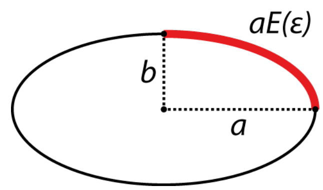

FIGURE 6. $aE(\varepsilon)$ is equal to one quarter of the circumference.

Solution by J. Pahikkala [17]. Let $f(\theta) := 1$ and $g(\theta) := \int_0^{\pi/2} \sqrt{1-\varepsilon^2 \sin^2\theta}\, d\theta$. Using the C-S inequality we get

$$\begin{aligned} \int_0^{\pi/2} \sqrt{1-\varepsilon^2\sin^2\theta}\, d\theta &\leq \sqrt{\int_0^{\pi/2} 1^2 d\theta} \sqrt{\int_0^{\pi/2} \left(1-\varepsilon^2\sin^2\theta\right) d\theta} \\ &= \sqrt{\frac{\pi}{2}} \sqrt{\int_0^{\pi/2} \left(1-\varepsilon^2 \frac{1-\cos 2\theta}{2}\right) d\theta} \\ &= \sqrt{\frac{\pi}{2}} \sqrt{\left[\theta - \varepsilon^2 \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^{\pi/2}} \\ &= \sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{\pi}{2} - \frac{\varepsilon^2\pi}{4}\right)} \\ &= \frac{\pi}{2}\sqrt{1-\frac{\varepsilon^2}{2}}, \end{aligned}$$

and as a result we have found an upper bound. □

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