Geometric Algebra for Electrical and Electronic Engineers

This tutorial paper provides a short introduction to geometric algebra, starting with its history and then presenting its benefits and exploring its applications.

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ABSTRACT | In this paper, we explicate the suggested benefits of Clifford's geometric algebra (GA) when applied to the field of electrical engineering. Engineers are always interested in keeping formulas as simple or compact as possible, and we illustrate that geometric algebra does provide such a simplified representation in many cases. We also demonstrate an additional structural check provided by GA for formulas in addition to the usual checking of physical dimensions. Naturally, there is an initial learning curve when applying a new method, but it appears to be worth the effort, as we show significantly simplified formulas, greater intuition, and improved problem solving in many cases.

KEYWORDS | Clifford algebra; Doppler effect; electromagnetism; geometric algebra (GA); Maxwell's equations; relativity

I. INTRODUCTION

Following Hamilton's invention of the quaternions in 1843 and his promotion of the idea that quaternions were a fundamental building block of the universe [1], Maxwell was inspired to write his set of electromagnetic equations in terms of the quaternions [2]. Unfortunately, it turned out that they were poorly suited to this task, and Heaviside—along with Gibbs and Helmholtz—was then motivated to develop a different system of vector analysis. This new system was readily adopted by both physicists and engineers due to its relative simplicity in comparison to quaternions [3].

Heaviside was to comment on his opposition to the use of quaternions in his book on electromagnetic theory: "I came later to see that, as far as the vector analysis I required was concerned, the quaternion was not only not required, but was a positive evil of no inconsiderable magnitude; and that by its avoidance the establishment of vector analysis was made quite simple and its working also simplified, and that it could be conveniently harmonized with ordinary Cartesian work" [4, p. 134].

Utilizing this vector notation, Heaviside was able to reduce Maxwell's ten field equations to the four equations now seen in modern textbooks [5] and shown below in SI units

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon} \quad \text{(Gauss' law)} \\ \nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \mu_0\mathbf{J} \quad \text{(Ampère's law)} \\ \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= \mathbf{0} \quad \text{(Faraday's law)} \\ \nabla\cdot\mathbf{B} &= 0 \quad \text{(Gauss' of magnetism)} \end{aligned} \tag{1}$$

where $\mathbf{E}$ is the electric field vector, $\mathbf{B}$ is the magnetic field vector, and $c=(\mu_0\epsilon_0)^{-1/2}$ is the speed of light in vacuum. However, while Heaviside achieved considerable success in encoding Maxwell's set of ten equations in just four equations (refer to Appendix A), looking at these four equations from a notational perspective, we can still identify several significant shortcomings.

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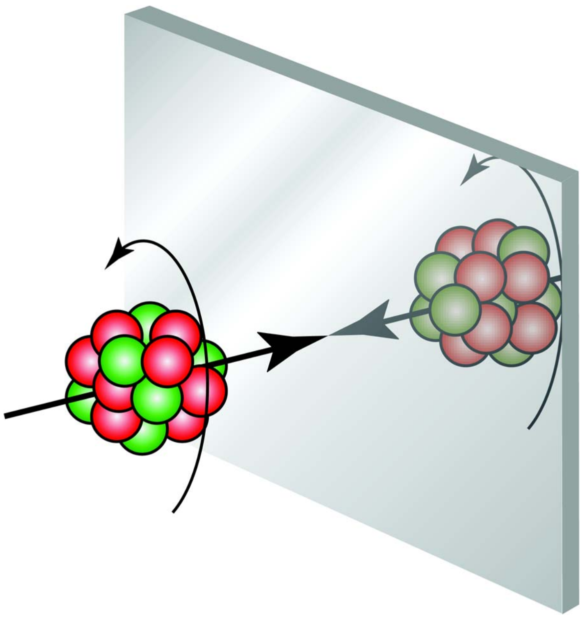

Fig. 1. ***Distinction between polar and axial vectors. It can be seen that the sense of rotation remains the same for the mirror image (axial vector). In contrast, the direction of motion of a particle, shown by the straight arrow, is inverted (polar vector). This illustrates the different natures of the $\mathbf{E}$ (polar) and $\mathbf{B}$ (axial) vectors. This ambiguity is resolved in geometric algebra (GA) where they are represented as vectors and bivectors, respectively.***

First, both the electric field $\mathbf{E} = (E_1, E_2, E_3)$ and the magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ are described by three component vectors. At first inspection, this notation appears reasonable as both fields are directional fields in a 3-D space. Investigating more thoroughly, however, we find that the magnetic field has different transformational properties than the electric field. This is normally taken into account through referring to the electric field as a polar vector and the magnetic field as an axial vector. This distinction is shown visually in Fig. 1 that shows under a reflection of the coordinate system that the electric field is inverted whereas the magnetic field is invariant. Hence, an ambiguity is introduced with the conventional Heaviside vector notation, in that, two quantities with different physical properties are both represented with the same mathematical object. As stated by Jackson "We see here . . . a dangerous aspect of our usual notation. The writing of a vector as '**a**' does not tell us whether it is a polar or an axial vector" [6, p. 270].

Second, as noted by Einstein, the relative strength of electric and magnetic fields depends on the relative speed of the observer. For a magnet being moved relative to a conductor, or the same conductor being moved relative to the magnet, the resulting current produced in the conductor is the same in both cases. However, for the case where the magnet is stationary clearly then there are only magnetic fields present, whereas for a moving magnet, using Faraday's law $\nabla \times \mathbf{E} = -(\partial \mathbf{B}/\partial t)$, then an electric field is assumed to be present. As Einstein commented: "The observable phenomena here depend only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases . . ." [7, p. 891]. Hence, rather than writing the electric and magnetic fields as independent vector objects, ideally we would have a single electromagnetic field variable that transforms in a consistent way depending on the relative motion. We will indeed find such an object through combining the electric and magnetic fields into a single field variable $\mathbf{E} + jc\mathbf{B}$, as detailed in Section II.

The third issue relates to the extensive use of complex numbers in electrical engineering theory [8]. Their widespread use is due to the fact that they allow a simple representation of sinusoidal waves and alternating current, as well as applications to complex permittivity and permeability, etc. However, complex numbers are not part of Cartesian space and so lack physical justification.

As the fourth and final issue, Maxwell's equations using Heaviside vector notation require four separate equations whereas most physical laws can be written as a single equation as in general relativity or Newton's law of universal gravitation, and so we would prefer to have a single equation describing electromagnetic effects.

The outline of the paper is as follows. After completing the main objective of the paper, of resolving four notational issues with the Gibbs' vector notation through the introduction of Clifford's geometric algebra (GA), we then go on to show other benefits of this approach when applied to calculation of areas and volumes as well as reflections and rotations. We then look at specific benefits of GA as applied to electromagnetism, such as dipoles, the electromagnetic potential, electromagnetic waves, the Liénard–Wiechert potential and circuit analysis. We then conclude with a final section showing how special relativity (SR) also naturally integrates with the formalism and how the ability in relativity to change reference frames provides new solution paths to electromagnetic problems.

II. DEFINITION OF CLIFFORD'S GEOMETRIC ALGEBRA

In 1637, Descartes developed the Cartesian coordinate system celebrated as one of the key mathematical developments in the progress of science that allows geometrical curves to be described algebraically and forms the foundation for both Heaviside's and Clifford's vector notation.

The heart of geometric algebra is to extend the vector space $\Re^n$ with an associative and anticommuting multiplication operation, along with new elements formed by the products of vectors. Letting e_1, e_2, and e_3 be an orthonormal basis for $\Re^n$, we first define

$$e_1{}^2 = e_2{}^2 = e_3{}^2 = 1 \tag{2}$$

fixing them as unit elements and where we now form 3-D vectors as linear combinations of these three elementary quantities. For example, the electric field vector can be represented as the object $\mathbf{E} = E_1e_1 + E_2e_2 + E_3e_3$. While this indeed succinctly expresses the three components of an electric field in 3-D space as a single object, as already discussed, it fails to provide a completely appropriate representation for the magnetic axial vector field.

We now proceed to expound Clifford's geometric algebra and see how it provides a solution to these notational dilemmas. In 1878, Clifford extended Descartes basic coordinate system in a straightforward way through simply allowing compound quantities to be formed from the three basis elements forming three bivector quantities e_1e_2, e_3e_1, and e_2e_3 as well as a trivector quantity $e_1e_2e_3$. We can then enforce orthogonality of the unit vectors with the rule that they anticommute, that is

$$e_1e_2 = -e_2e_1 \quad e_3e_1 = -e_1e_3 \quad e_2e_3 = -e_3e_2. \tag{3}$$

Defining the Clifford algebra through assuming an orthonormal basis[1] for $\Re^3$ allows us to view the basis elements e_1, e_2, e_3 as ordinary algebraic variables with additional properties such as a unit square and an anticommuting property rather than as Heaviside-type vectors that are subject to either the dot or cross products. This implies that mathematical manipulation of vector quantities is significantly simplified as it is now based on utilizing the well-known rules of elementary algebra [9], [10] rather than utilizing separately defined vector products.

Clifford's elegant generalization of Descartes' system also produces several other important advantages.

First, not only directed line segments—typically called vectors—but also oriented areas and oriented volumes can now be represented, that is, the bivectors e_1e_2, e_3e_1, and e_2e_3 and the trivector $e_1e_2e_3$, respectively. Furthermore, this expanded vector space, denoted $C\ell(\Re^3)$, contains all of the linear combinations of scalars, vectors, bivectors, and the trivector.

The second significant consequence of Clifford's idea is that the bivectors and the trivector square to minus one. Using the anticommutivity rule shown in (3), we find that $(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_2e_2e_1 = -e_1e_1 = -1$. It then follows that trivectors also square to minus one because $(e_1e_2e_3)^2 = e_1e_2e_3e_1e_2e_3 = -e_1e_2e_3e_1e_3e_2 = e_1e_2e_3e_3e_1e_2 = e_1e_2e_1e_2 = -1$. This is significant in that we can now provide a replacement for the unit imaginary. This has the advantage in that, while the conventional unit imaginary $\sqrt{-1}$ is purely abstract, the bivectors and trivectors of GA have a precise geometrical meaning in a Cartesian framework.

[1]While we have assumed an orthonormal Cartesian basis, Clifford's system can be readily extended to more general coordinate systems, such as nonorthogonal as well as polar, cylindrical, or spherical coordinates, for example. An example using nonorthogonal basis vectors with its associated reciprocal basis is shown in Appendix B.

We will now adopt the symbol j for the trivector quantity

$$j = e_1e_2e_3 \tag{4}$$

for the reasons that the trivector squares to minus one and commutes with all other elements—and so has the two key properties of the unit imaginary $\sqrt{-1}$. Hence, the symbol j can continue to be used in all electromagnetic calculations exactly as before, while its extra structure as a trivector can be further exploited. For example, if we are multiplying the trivector j with the basis element e_1, we find $je_1 = e_1e_2e_3e_1 = e_1^2e_2e_3 = e_2e_3$. This produces what are called the dual relations between vectors and bivectors

$$e_1e_2 = je_3 \quad e_3e_1 = je_2 \quad e_2e_3 = je_1. \tag{5}$$

This extra structure of the trivector is particularly useful in some contexts and not available with the conventional unit imaginary. Hence, within Clifford's system, complex-like numbers become a natural extension of physical Cartesian space rather than an *ad hoc* extension.

In order to assist the reader's intuition, we note an isomorphism with matrix algebra that $C\ell(\Re_3) \cong \mathrm{Mat}(2, C)$, where $C\ell(\Re^3)$ describes Clifford's vector system over $\Re^3$ that we are describing. This isomorphism also implies that Clifford algebra shares the noncommuting and associativity properties of matrix algebra. However, it should be noted that the Clifford algebra we have defined over $\Re^3$ has a greater degree of structure than the matrix definition, for example, we have a graded structure in $C\ell(\Re_3)$ of scalars, vectors, bivectors, and trivectors. It should be noted that we have chosen one particular approach in defining a Clifford algebra over $\Re^3$ through assuming an orthonormal basis e_1, e_2, e_3 but alternative[2] approaches are possible.

Regarding the product of two vectors within Clifford's system, we define two vectors as $\mathbf{v} = v_1e_1 + v_2e_2 + v_3e_3$ and $\mathbf{w} = w_1e_1 + w_2e_2 + w_3e_3$, and we find, using the law of the distribution of multiplication over additions (that is, expanding the brackets), the algebraic product of two vectors

$$\begin{aligned}\mathbf{vw} &= (v_1e_1 + v_2e_2 + v_3e_3)(w_1e_1 + w_2e_2 + w_3e_3)\\ &= v_1w_1 + v_2w_2 + v_3w_3 + (v_1w_2 - w_1v_2)e_1e_2\\ &\quad + (v_1w_3 - w_1v_3)e_1e_3 + (v_2w_3 - w_2v_3)e_2e_3\end{aligned} \tag{6}$$

[2]More generally, Clifford algebras $C\ell_n$ are an associative algebra with unity 1 of dimension 2^n where $n = \dim_{\Re} V$. These algebras can also be either simple, hence isomorphic to matrix algebras over the reals, complex numbers, or quaternions, or semisimple, isomorphic to the direct sum of two matrix algebras over the reals or quaternions. These algebras have been extensively classified and studied [13]–[16].

where we use the fact that the three unit elements square to unity and anticommute. In order to present it in a form that is more readily identifiable in terms of Heaviside notation we can use the dual relation, shown in (5), to write

$$\begin{aligned}\mathbf{vw} &= v_1w_1 + v_2w_2 + v_3w_3 + j(v_1w_2 - w_1v_2)e_3 \\ &\quad - j(v_1w_3 - w_1v_3)e_2 + j(v_2w_3 - w_2v_3)e_1 \\ &= \mathbf{v}\cdot\mathbf{w} + j\mathbf{v}\times\mathbf{w}. \end{aligned} \tag{7}$$

Hence, we find that the algebraic product of two Clifford vectors produces a combination of the dot and cross products[3] into a single complex-like number. This equation also illustrates the fact that parallel vectors commute and perpendicular vectors anticommute. From (6), we define the last three terms as the components of the wedge product, that is

$$\begin{aligned}\mathbf{v}\wedge\mathbf{w} = (v_1w_2 - w_1v_2)e_1e_2 &+ (v_1w_3 - w_1v_3)e_1e_3 \\ &+ (v_2w_3 - w_2v_3)e_2e_3. \end{aligned} \tag{8}$$

Hence, from (6) and (7), we can write the following relation:

$$\mathbf{v}\wedge\mathbf{w} = j\mathbf{v}\times\mathbf{w}. \tag{9}$$

The expression in (7), generated by expanding the brackets defining two vectors, thus provides an alternative calculation tool to the conventional method of using the determinant of two vectors embedded in a 3×3 matrix. Hence, there is no need to construct special definitions for products of vectors, such as the dot and cross products, as they both follow naturally from a straightforward algebraic expansion of Clifford vectors [11]–[13].

The wedge product is, in fact, more general than the cross product and is to be preferred for several reasons. First, the wedge product is associative and is easily extendable to any dimension, whereas the cross product[4] is nonassociative and is essentially only defined for three dimensions. The greater generality is provided because the wedge product is defined within the plane of the two vectors being multiplied, whereas the cross product requires an orthogonal vector, as shown in Fig. 2. Hence, in two dimensions, we can now define axial vector quantities such as angular velocity $j\boldsymbol{\omega} = \mathbf{r}\wedge\mathbf{v}$ or torque $j\boldsymbol{\tau} = \mathbf{r}\wedge\mathbf{F}$

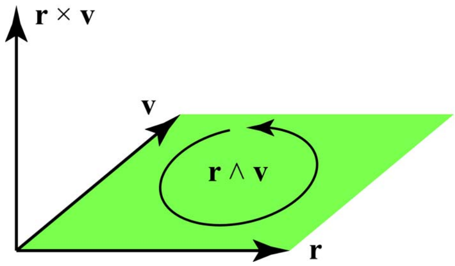

Fig. 2. ***The distinction between the cross product and the wedge product. While the cross and wedge products have the same scalar magnitude, calculated from (12) for the wedge product, i.e.,*** $\|\mathbf{r}\times\mathbf{v}\| = \|\mathbf{r}\|\|\mathbf{v}\|\sin\theta = |\mathbf{r}\wedge\mathbf{v}|$***, where*** θ ***is the angle between the two vectors, the wedge product describes an oriented plane defined by the two vectors, and so applies for any two vectors in any number of dimensions. The cross product, however, requires an orthogonal direction, and so is not defined in two dimensions and in four dimensions and higher there is an ambiguity between an infinity of possible orthogonal vectors.***

using the wedge product, whereas the cross product is technically not defined in this 2-D situation.

As can be seen from (6), for the case of a vector multiplied by itself, the wedge product will be zero and hence the square of a vector $\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v}$ becomes a scalar quantity. Hence, the Pythagorean length or norm of a vector is $\|\mathbf{v}\| = \sqrt{\mathbf{v}^2} = \sqrt{v_1^2 + v_2^2 + v_3^2}$. This now allows us to define the inverse of a vector $\mathbf{v}$ as

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v}^2} \tag{10}$$

where the inverse will fail to exist for $\mathbf{v} = 0$. Checking this result we find $\mathbf{vv}^{-1} = \mathbf{vv}/\mathbf{v}^2 = 1$ as required.

Now, because in Clifford algebra we are dealing with purely algebraic elements, we can form compound quantities called multivectors such as

$$M = a + \mathbf{v} + j\mathbf{w} + jb \tag{11}$$

where a and b are real scalars, $\mathbf{v}$ and $\mathbf{w}$ are vectors, and j is the trivector. Therefore, more generally, we can also define the inverse of a multivector as

$$M^{-1} = \frac{\bar{M}}{M\bar{M}} \tag{12}$$

where we define the conjugation operation $\bar{M} = a - \mathbf{v} - j\mathbf{w} + jb$, that is, inverting the sign of the vector and bivector components. We can then define the magnitude of a multivector as $|M| = \sqrt{M\bar{M}}$.

[3]We are working with the Clifford algebra $C\ell(\Re^3)$ over the reals, and so we adopt the number 1 as the unity element for the algebra. That is, in (7), we have implicitly assumed that $\mathbf{v}\cdot\mathbf{w}$ is defined as $(\mathbf{v}\cdot\mathbf{w})1$.

[4]It is possible to define an analogous vector cross product in seven dimensions though it fails to satisfy the Jacobi identity [13].

Thus, Clifford's system of vectors allows a general inverse operation to be defined. Hence, we can now divide by vectors; an operation that is not possible with either the dot or the cross product of Heaviside vectors. The reason for the noninvertibility of the dot product is that it produces a single scalar implying a loss of information. For a general cross product $\mathbf{u} = \mathbf{v} \times \mathbf{w}$, there are an infinite number of vectors[5] $\mathbf{w}$—lying within the plane perpendicular to $\mathbf{u}$—that gives the same resultant vector $\mathbf{u}$, and hence there is no unique inverse. The Clifford geometric product, on the other hand, as shown in (7), consisting of a combined dot and cross products, in general does have a unique inverse and so vector expressions can now be inverted.

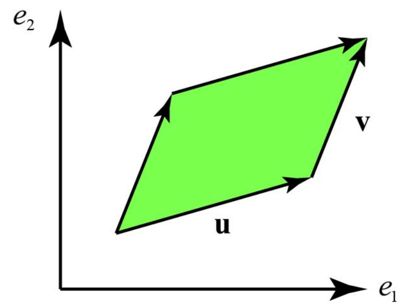

Fig. 3. *Calculating areas using the geometric product* $A = \langle \mathbf{uv} \rangle_2$.

A. Notation

In this paper, we attempt to use notation that is as close as possible to current usage in electrical engineering, while still seeking to faithfully represent quantities in the formalism of Clifford's geometric algebra. As the first principle, we write all vectors in bold font, and in accord with normal usage they can be either uppercase or lower case font, such as the electric field $\mathbf{E}$ or velocity $\mathbf{v}$, for example. Vectors are the only quantities that are given bold face so that there can be no confusion with other types of quantities. Bivectors are written as the trivector multiplied by a vector, as in the magnetic field bivector $j\mathbf{B}$, for example. Technically, the three basis vectors e_1, e_2, e_3 should be bold font and being constant should also be in upright Roman font, however, for the sake of readability, we write them as ordinary algebraic variables, as shown in italic unbolded font [11]. The trivector $j = e_1e_2e_3$ is also written in italic font for the same reasons of readability. All scalars are in lower case font, whether Latin or Greek characters, for example, the electric potential ϕ and the field energy u. Complex numbers are not used in Clifford's system—being replaced with algebraic quantities such as the trivector j—and so all scalars are therefore real numbers. All uppercase letters that are not bold are composite multivector quantities found in GA, such as the electromagnetic field $F = \mathbf{E} + jc\mathbf{B}$ or electromagnetic sources $J = \rho/\epsilon_0 - c\mu_0\mathbf{J}$. We also represent phasors with uppercase letters and tilde, such as $\widetilde{V}$, $\widetilde{I}$, and $\widetilde{R}$.

We believe this allows a clear and self-consistent set of notation when using GA in electromagnetism, that is mostly compatible with current usage and consistent with international standards.

III. AREAS AND VOLUMES

For two vectors $\mathbf{u} = u_1e_1 + u_2e_2$ and $\mathbf{v} = v_1e_1 + v_2e_2$, shown in Fig. 3, we might wish to know the area swept out by these two vectors. This can be calculated, using a variety of geometrical constructions, to be $u_1v_2 - u_2v_1$.

Now, from (7), we can write the product of the two vectors

$$\begin{aligned}\mathbf{uv} &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}\\ &= u_1v_1 + u_2v_2 + (u_1v_2 - u_2v_1)e_1e_2.\end{aligned} \tag{13}$$

We can see that the required area is the coefficient of the bivector term e_1e_2 that is produced from the wedge product $\mathbf{u} \wedge \mathbf{v}$. The bivector e_1e_2 represents a unit area as mentioned earlier, and so it is natural to expect this component to represent an area. Therefore, we can write for the enclosed area

$$A = \langle \mathbf{uv} \rangle_2 = (u_1v_2 - u_2v_1)e_1e_2 \tag{14}$$

where the notation $\langle \mathbf{uv} \rangle_2$ means to retain the second grade or bivector terms and discard the rest. Also, dimensionally, this makes sense because we are looking for a result with dimensions of area[6] or squared length. This dimensional argument also applies to three dimensions, where the volume will, therefore, need to be grade 3, that is, for a set of three vectors we find the enclosed volume $V = \langle \mathbf{uvw} \rangle_3$ as expected. Thus, a routine calculation of the algebraic product, followed by the selection of the desired components dimensionally, allows the relevant information to be extracted. For the special case where the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually orthogonal, we can dispense with the grade selection and simply write for the volume $V = \mathbf{uvw}$. Note that the value of the area and volume calculated in this way can return a positive or negative value that refers to their two possible orientations in space.

These examples serve to illustrate the natural way that GA models the geometry 3-D physical space, as shown in Fig. 4.

[5]The vector cross product can be written as $\mathbf{u} = \hat{u}|\mathbf{v}||\mathbf{w}|\sin\theta$, where $\hat{u}$ is the unit vector in the direction of $\mathbf{u}$ and θ is the angle between the two vectors. Hence, provided the product $|\mathbf{w}|\sin\theta$ is kept fixed, $|\mathbf{w}|$ and $\sin\theta$ can be independently varied, allowing an infinite number of vectors $\mathbf{w}$ producing the same cross product result $\mathbf{u}$.

[6]The wedge product, in general, defines the area swept out by a vector moving along a second vector [9].

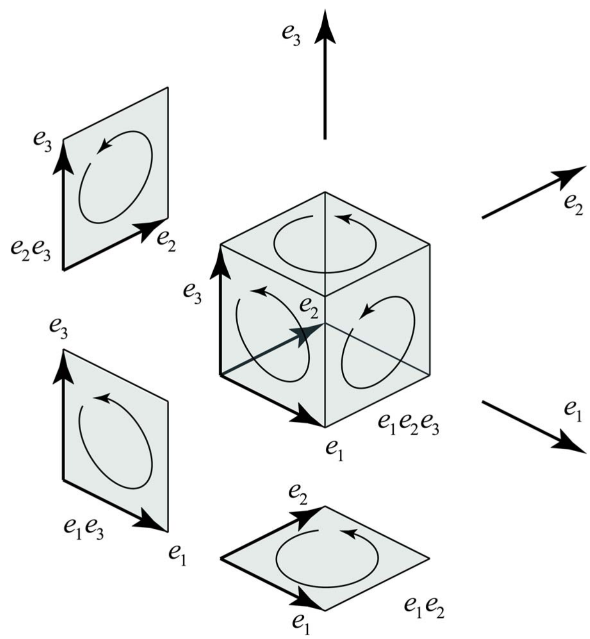

Fig. 4. ***The geometry of 3-D space modeled with Clifford's geometric algebra. In Clifford's system, 3-D space consists of linear combinations of lines, areas, and volumes. The three orthogonal lines represent Cartesian vectors, and the three orthogonal planes are isomorphic to Hamilton's quaternions, and the volume element producing complex-like numbers, thus unifying these three systems.***

IV. GENERALIZING COMPLEX NUMBERS TO INCLUDE CARTESIAN VECTORS

Complex numbers have been found useful in a wide range of engineering applications, such as for phasors and complex refractive indices, for example. They have also been combined with Cartesian vectors to represent electromagnetic waves, as in the Jones vector formalism, where the vector is used to represent the polarization direction and the complex number to contain the phase [17]. While this is effective in modeling polarization, it is a somewhat *ad hoc* construction, and so it is preferable to use GA that more naturally integrates complex-like numbers and Cartesian vectors. In 2-D, the full Clifford space $C\ell(\Re^2)$ can be represented as the set of elements

$$a + v_1 e_1 + v_2 e_2 + b e_1 e_2 \tag{15}$$

where a, v_1, v_2, b are real-valued scalars. This Clifford algebra turns out to be isomorphic to the matrix algebra $C\ell(\Re^2) \cong \mathrm{Mat}(2, \Re)$. We find that the even subalgebra $a + b e_1 e_2$ is isomorphic to the complex numbers[7] and $v_1 e_1 + v_2 e_2$ is a representation for Cartesian vectors, and so both types of quantities can be represented within the same space.

One of the main properties of complex numbers is when represented on an Argand diagram and multiplied by the unit imaginary they experience a rotation of $\pi/2$ radians. We can duplicate this property of the unit imaginary using the bivector of the plane e_1e_2. For example, to rotate the Cartesian vector ae_1 (lying on the e_1-axis) through $\pi/2$ radians, we can multiply from the right with the bivector e_1e_2. That is, $ae_1(e_1e_2) = ae_2$, which is a $\pi/2$ rotation as required. Multiplying from the left will produce a rotation by $-\pi/2$. More generally, any complex number on an Argand diagram can be rotated by an angle θ, through acting with the operator $\mathrm{e}^{\theta\sqrt{-1}}$. We can write this in GA, for a Cartesian vector $v = v_1e_1 + v_2e_2$, as

$$\mathbf{v}' = \mathrm{e}^{\theta e_1 e_2}\mathbf{v} = (\cos\theta + e_1 e_2 \sin\theta)(v_1 e_1 + v_2 e_2). \tag{16}$$

While isomorphic to the rotation of complex numbers on the Argand plane, this formula allows us to rotate real Cartesian vectors while still utilizing the efficient rotation properties similar to the unit imaginary that we duplicate with the bivector of the plane.

The use of (16) that rotates Cartesian vectors in 2-D also helps elucidate Euler's intriguing mathematical formula $\mathrm{e}^{\pi\sqrt{-1}} = -1$. Equation (16) shows that rotating a Cartesian vector $\mathbf{v}$ by π radians produces $\mathbf{v}' = -\mathbf{v}$, or a flip in direction, thus enabling Euler's formula to be demonstrated on the real Cartesian plane.

V. ELECTROMAGNETISM IN CLIFFORD NOTATION

Now, equipped with this basic knowledge of GA, we can address specifically some of the notational problems with Heaviside vectors described earlier. The first notational problem, regarding the correct representation of the magnetic field as an axial vector, can be solved by representing this quantity not as a traditional vector but as a three-component bivector

$$\begin{aligned} B_1e_2e_3 + B_2e_3e_1 + B_3e_1e_2 &= e_1e_2e_3(B_1e_1 + B_2e_2 + B_3e_3) \\ &= j\mathbf{B} \end{aligned} \tag{17}$$

that now has the required transformational properties, which is immediately clear from the notation. That is, if we represent the polar vector $\mathbf{E} = E_1e_1 + E_2e_2 + E_3e_3$, then if we invert the orientation of the coordinate system (parity transform) through $e_1 \to -e_1, e_2 \to -e_2, e_3 \to -e_3$, then we see that $\mathbf{E} \to -\mathbf{E}$ whereas $\mathbf{B} \to \mathbf{B}$ giving the required

[7] The subalgebra of $C\ell_2$ spanned by 1 and e_1e_2 is isomorphic to $\mathbb{C}$, however, unlike $C\ell(\Re^3)$, the element e_1e_2 is not commuting with other elements of the algebra and so does not belong to the center $\mathrm{Cen}(C\ell_2)$.

Table 1 A Periodic Table of Physical Quantities Used in Electromagnetism Categorized According to Their Scalar, Vector, Bivector, or Trivector Nature. We Have the Classical Electromagnetic Force Vector $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$, With the Vector Field $\mathbf{E} = -\nabla\phi - (\partial\mathbf{A}/\partial t)$, Bivector Field $j\mathbf{B} = \nabla \wedge \mathbf{A}$, and the Poynting Vector $\mathbf{S} = (1/\mu_0)j\mathbf{B} \wedge \mathbf{E}$

Scalar	Vector	Bivector	Trivector
	Electric field $\mathbf{E}$	Magnetic field $j\mathbf{B}$	Helicity $\mathbf{A} \wedge \nabla \wedge \mathbf{A}$
	Electric dipole $\mathbf{p}$	Magnetic dipole $j\mathbf{m}$	
	Displacement $\mathbf{x}$	Area $\mathbf{x}_1 \wedge \mathbf{x}_2$	Volume $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$
	Velocity $\mathbf{v}$	Angular velocity $\mathbf{x} \wedge \mathbf{v}$	
Field energy u	Poynting vector $\mathbf{S}$		
Mass m	Force $\mathbf{F}$	Torque $\mathbf{x} \wedge \mathbf{F}$	
Electric Potential ϕ	Magnetic potential $\mathbf{A}$		
Electric charge ρ	Electric current $\mathbf{J}$	Magnetic monopole current $\mathbf{J}^m$	Magnetic monopole charge ρ^m

distinct transformational properties[8] between the electric and magnetic fields. This notational improvement also allows us to remove the second notational defect in that we can now represent the electromagnetic field as a single field variable $F = \mathbf{E} + jc\mathbf{B}$. In order to combine two quantities like this, clearly, the units must agree and we can achieve this by multiplying $\mathbf{B}$ by the speed of light c because $\mathbf{B}$ has dimensions of force per charge per unit velocity. This representation of the electromagnetic field is a considerable simplification over tensor notation that requires a 4×4 antisymmetric matrix in order to provide a similar unification, as shown in (66).

Now that we have the ability to appropriately represent axial vectors as bivectors we can now catalog these types of vectors found in electromagnetism. In 3-D, as well as being able to identify the distinction between polar and axial vectors, we can also identify scalar as well as trivector quantities. Hence, these four types of algebraic quantities can be used to categorize the various physical quantities found in engineering in the form of a "periodic" table, as shown in Table 1.

We can now first inspect the list of vector type quantities and recognize linear motion such as velocity, momentum, acceleration (including force), and the electric field as vectors (polar), while rotational type quantities, angular velocity, torque, and the magnetic field as bivector (axial) type quantities. Using the conventional formula, the Poynting vector $\mathbf{S} = (1/\mu_0)\mathbf{E} \times \mathbf{B}$, while at first sight appears to be an axial or bivector-like quantity due to the presence of the cross product, in fact is a vector quantity. This situation is clarified in GA when we realize that the magnetic field is a bivector not a vector quantity as assumed in the conventional formula for $\mathbf{S}$. Indeed, in GA, we write for the Poynting vector $\mathbf{S} = (1/\mu_0)j\mathbf{B} \wedge \mathbf{E}$, which under a parity transform now correctly produces $\mathbf{S} \to -\mathbf{S}$. The same issue arises in the Lorentz force law $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where because the magnetic field is a bivector quantity and not a vector one, the magnetic force component $\mathbf{v} \times \mathbf{B}$ will be a polar vector rather than an axial one. Writing the Lorentz force in GA as $q(\mathbf{E} + j\mathbf{B} \wedge \mathbf{v})$, we see that the magnetic force now transforms as a polar vector, as required.

This insight that the magnetic field is a bivector or areal quantity as opposed to a linear vector quantity also answers a common question regarding the Lorentz force law. Why is the magnetic force not in the direction of the magnetic field vector as it is for the electric field vector? The answer is that the Lorentz force for the magnetic field lies in the plane defined by the bivector magnetic field $j\mathbf{B}$, and orthogonal to the velocity. The actual direction of deflection in the plane of the magnetic field will be given by one of the two possible orientations of the bivector field. This will turn out to be the direction of deflection conventionally calculated with the right-hand side rule.

Table 1 also has a column for scalar quantities that contains the nondirectional quantities such as energy and mass as well as the electric potential. The vector potential is a vector quantity as expected, but of interest is the fact that magnetic monopole currents and charges appear as bivectors and trivectors, respectively. While magnetic monopoles have never been confirmed to exist in nature, it is interesting in theory to note their intrinsic trivector nature, as explored further in Appendix C. If we consider other physical quantities that might have a trivector nature, we can identify the magnetic flux and the helicity of magnetic field lines as further examples. These examples serve to illustrate the extra level of verification that can be applied to formulas with reference to the four types of quantities. For example, if we are calculating the angular momentum of an object, then we would expect a bivector result from our calculations.

We also need to keep in mind that some physical quantities, such as the electromagnetic field, for example, are composite quantities consisting of both vector and

[8]First, "It is an experimental fact that electric charge is invariant under Galilean and Lorentz transformations and is a scalar under rotations ... charge is also a scalar under spatial inversion ..." [6]. Using this as a starting point, it then follows from the Maxwell equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ that $\mathbf{E}$ is a polar vector as both sides must transform in the same way. Then, Faraday's law $\nabla \times \mathbf{E} + (\partial\mathbf{B}/\partial t) = 0$ implies that $\mathbf{B}$ is an axial vector, due to the presence of the cross product. Note that for a vector constructed from the cross product of two other vectors $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, then under spatial inversion $\mathbf{w}' = (-\mathbf{u}) \times (-\mathbf{v}) = \mathbf{u} \times \mathbf{v} = \mathbf{w}$, and so $\mathbf{w}$ is invariant.

Table 2 Composite Electromagnetic Variables. The Field Is a Composite Quantity $F = \mathbf{E} + jc\mathbf{B}$, With the Field Potential $A = \phi - c\mathbf{A}$ and a Source $J = \rho/\epsilon_0 - c\mu_0\mathbf{J}$. We Have Used $\partial = (1/c)(\partial/\partial t) + \nabla$ and Velocity Multivector $V = c - \mathbf{v}$. We Have $\partial F = J$ as Well as the Conservation Law of Charge $\partial \cdot J = 0$, Lorenz Gauge $\partial \cdot A = 0$, and the Conservation of Energy $\partial \cdot U = 0$

Electromagnetic theory	Clifford multivector	Comments
Electromagnetic Potential	$A = \phi - c\mathbf{A}$	Scalar + Vector, gauge freedom $A \rightarrow A - \partial\lambda$
Electromagnetic Field	$F = \mathbf{E} + jc\mathbf{B}$	Vector + Bivector, $F = \bar{\partial}A = \left(\frac{1}{c}\frac{\partial}{\partial t} - \nabla\right)A$
Field Energy	$U = \frac{1}{2}\epsilon_0 F F^\dagger = u + \mathbf{S}/c$	Scalar + Vector, $u = \frac{1}{2}\epsilon_0(\mathbf{E}^2 + c^2\mathbf{B}^2)$, $\mathbf{S} = \frac{1}{\mu_0} j\mathbf{B} \wedge \mathbf{E}$
Invariants	$F^2 = \mathbf{E}^2 - c^2\mathbf{B}^2 + 2jc\mathbf{B} \cdot \mathbf{E}$	Scalar + Trivector, Lagrangian density
Force	$cK = qVF$	Scalar+Vector+Bivector+Trivector, Vector component represents the force
Sources	$J = \rho/\epsilon_0 - c\mu_0\mathbf{J}$	Scalar + Vector

bivector components. A list of these composite quantities is shown in Table 2, which includes the field potential $A = \phi - c\mathbf{A}$ and an electromagnetic source $J = \rho/\epsilon_0 - c\mu_0\mathbf{J}$. A second useful transformation on the multivector M shown in (11), in addition to Clifford conjugation, is called reversion and defined as $M^\dagger = a + \mathbf{v} - j\mathbf{w} - jb$ that inverts the sign of the bivector and trivector components. Both of these operations are antiautomorphisms, that is, $(MN)^\dagger = N^\dagger M^\dagger$. If we apply both operations, we produce space inversion $\bar{M}^\dagger = a - \mathbf{v} + j\mathbf{w} - jb$. The use of a single multivector variable to represent a set of related quantities, such as J to represent electromagnetic sources, will be shown to lead to more compact notation for many equations, and a comparison of the conventional Heaviside–Gibbs vector notation and Clifford's GA for a range of electromagnetic equations is shown in Table 3.

The Poynting vector in electromagnetism is typically defined as $\mathbf{S} = (1/\mu_0)\mathbf{B} \times \mathbf{E}$. However, in a true 2-D space, the cross product is not defined, although clearly a Poynting energy flow must still exist. This is typically dealt with by assuming an embedding in 3-D so that we then have access to an orthogonal direction that can be used to represent the magnetic field. However, we would prefer to be able to calculate the Poynting vector dealing only with elements defined within the plane itself. Now, in GA, the electromagnetic field is represented as $F = \mathbf{E} + jc\mathbf{B}$, where for an electromagnetic field in 2-D, we have $F = E_1e_1 + E_2e_2 + jce_3B = E_1e_1 + E_2e_2 + e_1e_2cB$, which is defined only in terms of planar elements. The Poynting vector is also well defined as $\mathbf{S} = (1/\mu_0)j\mathbf{B} \wedge \mathbf{E} = (1/\mu_0)e_1e_2B\mathbf{E}$. As can be seen, this equation only uses elements defined within the plane, the elements e_1e_2, the scalar B, and the planar vector $\mathbf{E} = E_1e_1 + E_2e_2$, thus allowing the electromagnetic field and the Poynting vector to be defined with only planar elements. This approach could, therefore, be utilized when analyzing coplanar waveguides, for example.

Now we address the third issue, regarding the forming of a single field equation for electromagnetism. Inspecting the result of the product of two Clifford vectors we can see that if we define a gradient vector $\nabla = e_1(\partial/\partial x) + e_2(\partial/\partial y) + e_3(\partial/\partial z)$, then we have the algebraic product

Table 3 Comparison of the Heaviside-Gibbs Vector Notation Versus Clifford's Notation, Where We Have Used the Potential Multivector $A = \phi - c\mathbf{A}$, Velocity Multivector $V = c - \mathbf{v}$, and Source Multivector $J = (\rho/\epsilon) - c\mu_0\mathbf{J}$

EM theory	Heaviside-Gibbs vectors	Clifford's GA
Field	$\mathbf{E},\mathbf{B}$	$F = \mathbf{E} + jc\mathbf{B}$
EM equations	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}, \nabla \cdot \mathbf{B} = 0$	$\partial F = J$
	$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0\mathbf{J}$	
	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	
Charge conservation	$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$	$\partial \cdot J = 0$
Energy in fields	$u = \frac{1}{2}\epsilon_0(\mathbf{E}^2 + c^2\mathbf{B}^2)$	$U = -\frac{1}{2}\epsilon_0 F F^\dagger$
Poynting vector	$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$	
Conservation energy	$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0$	$\partial \cdot U = 0$
Invariants	$c^2\mathbf{B}^2 - \mathbf{E}^2$, $\frac{2}{c}\mathbf{B} \cdot \mathbf{E}$	F^2
Action $E_p - E_k$	$\rho\phi - \mathbf{J} \cdot \mathbf{A}$	$J \cdot A$
Force	$\mathbf{K} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	$K = \langle \frac{q}{c} VF \rangle_1$
Potential function A	$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$	$F = \bar{\partial}A$
	$\mathbf{B} = \nabla \times \mathbf{A}$	
Lorenz gauge	$c\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$	$\partial \cdot A = 0$
Potential equations	$\partial^2 \mathbf{A} = -\mu_0\mathbf{J}$, $\partial^2\phi = -\frac{\rho}{\epsilon_0}$	$\partial\bar{\partial}A = J$

$\nabla\mathbf{E} = \nabla\cdot\mathbf{E} + j\nabla\times\mathbf{E}$, from (7). Hence, inspecting (1), we can see that we can add Maxwell's first and third equations, after multiplying through by j, and the second and fourth equations to give

$$\nabla\mathbf{E} + j\frac{\partial\mathbf{B}}{\partial t} = \frac{\rho}{\epsilon_0}$$
$$cj\nabla\mathbf{B} + \frac{\partial\mathbf{E}}{c\partial t} = -c\mu_0\mathbf{J} \quad (18)$$

which eliminates the need for the dot and cross products. These two equations can now be added to give

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)(\mathbf{E} + jc\mathbf{B}) = \frac{\rho}{\epsilon_0} - c\mu_0\mathbf{J} \quad (19)$$

thus combining Maxwell's four equations into a single equation [10], [18]. If we now define a four-gradient operator that includes time $\partial = (1/c)(\partial/\partial t) + \nabla$ and a source term combining charge and current into $J = (\rho/\epsilon) - c\mu_0\mathbf{J}$, along with a field variable $F = \mathbf{E} + jc\mathbf{B}$, Maxwell's equations are reduced to a single equation

$$\partial F = J. \quad (20)$$

This is, in fact, one of the simplest possible first-order differential equations that can be written. This now allows an intuitive interpretation of Maxwell's equations, namely that the gradient of the field F is proportional to the electromagnetic sources J that are present. This equation is also easily generalized to linear media, substituting $\mu_0 \rightarrow \mu$, $\epsilon_0 \rightarrow \epsilon$, and where the speed of transmission of electromagnetic signals is, therefore, modified to $c = 1/\sqrt{\mu\epsilon}$. Maxwell's equations in a completely general medium using GA require a more sophisticated treatment as found in [19].

Conventional relativistic treatments produce the equation $\partial_\mu F^{\mu\nu} = J^\nu$ for Maxwell's equations. While there is a superficial resemblance with (20), this equation uses tensor contraction of indices labeled μ equivalent to matrix multiplication of a four-vector and 4×4 matrix, where J^ν represents the four current. This can be compared with (20) and an expanded form in (19) that uses straightforward vector notation and elementary algebraic operations, thus providing more transparent notation for the engineer.

The four boundary conditions across an interface with surface normal $\hat{\mathbf{n}}$ are typically given by

$$\mathbf{E}_2^\perp - \mathbf{E}_1^\perp = \frac{1}{\epsilon_0}\sigma \quad \mathbf{E}_2^\parallel - \mathbf{E}_1^\parallel = 0$$
$$\mathbf{B}_2^\perp - \mathbf{B}_1^\perp = 0 \quad \mathbf{B}_2^\parallel - \mathbf{B}_1^\parallel = \mu_0\mathbf{K}\times\hat{\mathbf{n}} \quad (21)$$

where $\hat{\mathbf{n}}$ is the normal vector to a boundary defining perpendicular and parallel field components. Now, using a single field variable $\Delta F = \Delta\mathbf{E} + jc\Delta\mathbf{B}$, we can write

$$\Delta F = \hat{\mathbf{n}}K \quad (22)$$

where the surface current multivector $K = \sigma/\epsilon_0 - c\mu_0\mathbf{K}$. This provides a single easy to remember the expression to calculate the field discontinuities across an interface, as opposed to four separate equations. This equation can be interpreted as a discrete form of Maxwell's equations, shown in (20).

The fourth complaint we had was in regard to the extensive use of imaginary quantities in electromagnetism. We have already seen how in 2-D the unit imaginary can be replaced with the bivector of the plane e_1e_2 that allows us to continue to work in a real Cartesian space while still having the benefits of complex-like numbers. In 3-D, the unit imaginary can be replaced with the trivector $j = e_1e_2e_3$. Hence, rather than imaginary numbers being an *ad hoc* addition creating an unphysical complex space, we can now generate algebraic equivalents to the unit imaginary allowing us to stay within real physical Cartesian space.

Incidentally, we find an isomorphism between Hamilton's quaternions i, j, k and the three Clifford bivectors e_1e_2, e_3e_1, and e_2e_3. Hence, the basis of the dispute between the followers of Hamilton's quaternions and the Cartesian vectors of Heaviside can now be clarified, namely that Hamilton misinterpreted his three quaternions to be a basis for the three translational freedoms of physical space, whereas, in fact, they represented the three orthogonal rotational freedoms of 3-D space. Therefore, Clifford's vector system perhaps helps us resolve the dispute between the followers of Gibbs–Heaviside vectors and those of Hamilton's quaternions through identifying the Heaviside–Gibbs vectors with the three vectors e_1, e_2, e_3 and the three quaternions as the three bivectors e_1e_2, e_3e_1, e_2e_3. Hence, Clifford's systems allow us to unify vectors, quaternions and complex numbers into a unified real algebraic space $C\ell(\Re^3)$.

VI. APPLICATIONS OF GA TO ELECTROMAGNETISM

After covering the general topic of how reflections and rotations are handled in GA, we then present five applications of GA to electromagnetism: 1) dipoles; 2) the electromagnetic potential; 3) electromagnetic waves; 4) the Liénard–Wiechert potentials; and finally 5) circuit analysis and complex power [20], [21].

A. Reflection and Rotation of Vectors

Defining the bivector $\hat{N} = j\hat{\mathbf{n}}$ that represents a unit plane orthogonal to the unit vector $\hat{\mathbf{n}}$ and then defining the

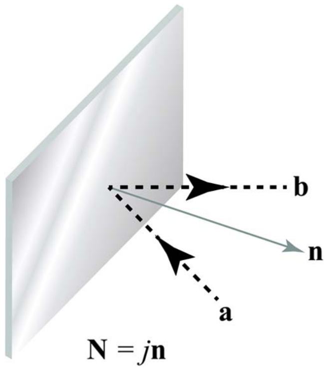

Fig. 5. *A light ray incident on a plane mirror* $\hat{N} = j\hat{\mathbf{n}}$. *We find the reflected ray* $\mathbf{b} = \hat{N}\mathbf{a}\hat{N}$.

following bilinear transformation:

$$\mathbf{v}' = \hat{N}\mathbf{v}\hat{N} = -\hat{\mathbf{n}}\mathbf{v}\hat{\mathbf{n}} \tag{23}$$

we produce a reflection of the vector $\mathbf{v}$ in the mirror plane $\hat{N}$, as shown in Fig. 5. This can be seen by writing $\mathbf{v} = \mathbf{v}^{\perp} + \mathbf{v}^{\parallel}$ as the sum of perpendicular and parallel components with respect to the unit vector $\hat{\mathbf{n}}$ and then we find

$$\mathbf{v}' = -\hat{\mathbf{n}}(\mathbf{v}^{\perp} + \mathbf{v}^{\parallel})\hat{\mathbf{n}} = \mathbf{v}^{\perp} - \mathbf{v}^{\parallel} \tag{24}$$

using the fact that perpendicular vectors anticommute ($\mathbf{v}^{\perp}\hat{\mathbf{n}} = -\hat{\mathbf{n}}\mathbf{v}^{\perp}$) and parallel vectors commute ($\mathbf{v}^{\parallel}\hat{\mathbf{n}} = \hat{\mathbf{n}}\mathbf{v}^{\parallel}$) and that $\hat{\mathbf{n}}^{\mathbf{2}} = \mathbf{1}$. Hence, the component of $\mathbf{v}^{\parallel}$ parallel to the vector $\hat{\mathbf{n}}$ (and perpendicular to the plane $j\hat{\mathbf{n}}$) is reflected, and the component of $\mathbf{v}^{\perp}$ perpendicular to the vector $\hat{\mathbf{n}}$ (and parallel to the plane $j\hat{\mathbf{n}}$) is unchanged. Therefore, (23) describes the reflection of a light wave of initial direction $\mathbf{v}$ from a plane mirror $\hat{N} = j\hat{\mathbf{n}}$; see Fig. 5.

If we now produce an additional reflection from a second mirror $\hat{M} = j\hat{\mathbf{m}}$ acting on the reflected light ray $\mathbf{v}'$, then we produce the vector

$$\mathbf{v}'' = \hat{M}\mathbf{v}'\hat{M} = \hat{M}\hat{N}\mathbf{v}\hat{N}\hat{M} = \hat{\mathbf{m}}\hat{\mathbf{n}}\mathbf{v}\hat{\mathbf{n}}\hat{\mathbf{m}}. \tag{25}$$

Now, using associativity we can inspect the product

$$\hat{\mathbf{m}}\hat{\mathbf{n}} = \hat{\mathbf{m}}\cdot\hat{\mathbf{n}} + \hat{\mathbf{m}}\wedge\hat{\mathbf{n}} = \cos\theta - \hat{\mathbf{B}}\sin\theta = \mathrm{e}^{-\hat{\mathbf{B}}\theta} \tag{26}$$

where θ is the angle between the two unit vectors, and the unit bivector, describing the plane of the two vectors $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$, is $\hat{B} = -(\hat{\mathbf{m}}\wedge\hat{\mathbf{n}}/|\hat{\mathbf{m}}\wedge\hat{\mathbf{n}}|) = -(\hat{\mathbf{m}}\wedge\hat{\mathbf{n}}/\sin\theta)$ with the property that $\hat{B}^2 = -1$. We use here the result that $|\hat{\mathbf{m}}\wedge\hat{\mathbf{n}}| = \|\hat{\mathbf{m}}\|\|\hat{\mathbf{n}}\|\sin\theta = \sin\theta$.

Substituting (26) back into (25), we find

$$\mathbf{v}'' = \mathrm{e}^{-\hat{B}\theta}\mathbf{v}\mathrm{e}^{\hat{B}\theta} \tag{27}$$

where we have opposite signs in the exponential due to the fact that for the wedge product $\hat{\mathbf{m}}\wedge\hat{\mathbf{n}} = -\hat{\mathbf{n}}\wedge\hat{\mathbf{m}}$. If we now split $\mathbf{v}$ into components parallel and perpendicular to the plane $\hat{B}$, then we find

$$\mathbf{v}'' = \mathrm{e}^{-\hat{B}\theta}(\mathbf{v}^{\perp} + \mathbf{v}^{\parallel})\mathrm{e}^{\hat{B}\theta} = \mathbf{v}^{\perp} + \mathbf{v}^{\parallel}\mathrm{e}^{2\hat{B}\theta}. \tag{28}$$

We see that the perpendicular component to the plane $\hat{B}$ is unchanged, and the parallel component to the plane is rotated by an angle 2θ. We can see that the operation $\mathrm{e}^{2\hat{B}\theta}$ rotates the vector $\mathbf{v}^{\parallel}$ by an angle 2θ by analogy with rotations in the Argand plane, as both $\hat{B}$ and $\mathbf{v}^{\parallel}$ lie in the same plane. This behavior is, in fact, exactly what we require of a rotation in 3-D, that is, the parallel components to the plane are rotated and the perpendicular components unaffected. Hence, we can conclude that to rotate a vector $\mathbf{v}$ by an angle θ using the rotation plane $\hat{B}$, we act with

$$\mathbf{v}' = \mathrm{e}^{-\hat{B}\theta/2}\mathbf{v}\mathrm{e}^{\frac{\hat{B}\theta}{2}} = \mathrm{e}^{-j\hat{\mathbf{a}}\theta/\mathbf{2}}\mathbf{v}\mathrm{e}^{\frac{j\hat{\mathbf{a}}\theta}{2}}. \tag{29}$$

We have included the dual form of the rotation operation that utilizes the unit vector $\hat{\mathbf{a}}$ that is perpendicular to the plane $\hat{B} = j\hat{\mathbf{a}}$. This can then be interpreted as rotating the vector $\mathbf{v}$ about the axis $\hat{\mathbf{a}}$ by an angle θ. This derivation confirms the well-known result that two reflections create a rotation [11]. We can thus write this equation as

$$\mathbf{v}' = R\mathbf{v}R^{\dagger} = \mathrm{e}^{-j\mathbf{a}/2}\mathbf{v}\mathrm{e}^{\frac{j\mathbf{a}}{2}} \tag{30}$$

with the rotation operator $R = \mathrm{e}^{-j\mathbf{a}/2}$. The bivector $j\mathbf{a}$ sets the plane of rotation, with a perpendicular axis $\mathbf{a}$, that rotates all vectors $\theta = \|\mathbf{a}\|$ radians within this plane.

B. Dipoles

We have an electric dipole moment $\mathbf{p} = qd\mathbf{r}$ that is categorized as a polar vector and the magnetic moment $\mathbf{m} = Id\mathbf{A}$, where I is a current around a loop of cross section A. We know bivectors represent unit areas, so as anticipated the magnetic dipole is a bivector. Hence, for a charge configuration consisting of both electric and magnetic dipoles, we can write a dipole multivector

$$Q = \mathbf{p} - j\mathbf{m} \tag{31}$$

acting on a field F, we can form the product, and retaining just the scalar and bivector terms of interest, we find

$$\begin{aligned}\langle QF\rangle_{02} &= \langle(\mathbf{p}-j\mathbf{m})(\mathbf{E}+jc\mathbf{B})\rangle_{02}\\ &= (\mathbf{p}\cdot\mathbf{E}+c\mathbf{m}\cdot\mathbf{B})+(\mathbf{p}\wedge\mathbf{E}+c\mathbf{m}\wedge\mathbf{B})\end{aligned} \tag{32}$$

where the first two terms give the stored energy and the second two terms give the applied torque.

For example, for a compass needle direction, denoted by the dipole multivector $Q=-j\mathbf{m}$, oriented in the Earth's magnetic field vector $F=jc\mathbf{B}$, we find

$$QF = c\mathbf{m}\cdot\mathbf{B}+c\mathbf{m}\wedge\mathbf{B} = c\mathbf{m}\cdot\mathbf{B}+jc\mathbf{m}\times\mathbf{B} \tag{33}$$

where the scalar represents the stored energy and the bivector represents the applied torque. The units also match as required as both energy and torque are measured in Joules.

C. Potential Formulation in GA

Defining a multivector potential

$$A=\phi-c\mathbf{A} \tag{34}$$

where $\mathbf{A}$ is the vector potential, we find

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t}-\nabla\right)(\phi-c\mathbf{A}) &= \left(\frac{1}{c}\frac{\partial\phi}{\partial t}+c\nabla\cdot\mathbf{A}\right)+\left(-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}\right)\\ &\quad + c\nabla\wedge\mathbf{A}\\ &= l+\mathbf{E}+jc\mathbf{B}\end{aligned} \tag{35}$$

where we have made the correspondence $\mathbf{E}=-\nabla\phi-(\mathbf{A}/\partial t)$ and $j\mathbf{B}=\nabla\wedge\mathbf{A}=j\nabla\times\mathbf{A}$ with $l=(1/c)\times(\partial\phi/\partial t)+c\nabla\cdot\mathbf{A}$. If we set $l=0$, then we are adopting the Lorenz gauge. Coincidentally, this is the unique gauge that puts the potentials and the fields onto a causal basis with the sources. We thus can write $F=\bar{\partial}A$, where $\bar{\partial}=(1/c)(\partial\phi/\partial t)-\nabla$. Substituting this into Maxwell's equation in (20), we produce

$$\partial\bar{\partial}A=\left(\partial_{ct}^2-\nabla^2\right)A=J. \tag{36}$$

This naturally splits into four copies of Poisson's equation that have known solutions

$$A=\frac{\mu_0}{4\pi}\int_{\mathrm{Vol}}\frac{J'}{r'}d\tau' \tag{37}$$

where the primes indicate that we evaluate the influence of the sources at a retarded time, which allows for the propagation of electromagnetic effects at the speed of light c. This solution, of course, is known but we have the advantage that we can provide the solution to Maxwell's equation in a single formula not needing to be split into separate equations for the electric and magnetic potential. The field is then recovered from $F=\bar{\partial}A$.

D. EM Wave

The field variable $F=\mathbf{E}+jc\mathbf{B}$ in general can describe electromagnetic fields varying in both space and time and so can be used to describe electromagnetic wave propagation. For a plane electromagnetic wave, we will describe an initial field as $F_0=\mathbf{E}_0+jc\mathbf{B}_0$ defined at some point in space and time, selected here as $\mathbf{r}=0$ and $t=0$. Now, for a sinusoidal variation in this field, propagating in the direction $\mathbf{k}$ from this point, we can write

$$F=F_0\mathrm{e}^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)}=F_0(\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)+j\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)) \tag{38}$$

where ω is the angular frequency of the sinusoidal variation. We should first check that this indeed satisfies Maxwell's source free equations, finding

$$\partial F=\left(\frac{1}{c}\frac{\partial}{\partial t}+\nabla\right)F_0\mathrm{e}^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)}=\left(-\frac{\omega}{c}+\mathbf{k}\right)F_0\mathrm{e}^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{39}$$

For a source free solution $\partial F=0$, we, therefore, require

$$\begin{aligned}0&=\left(-\frac{\omega}{c}+\mathbf{k}\right)(\mathbf{E}_0+jc\mathbf{B}_0)\\ &=\mathbf{k}\cdot\mathbf{E}_0-\left(\frac{\omega\mathbf{E}_0}{c}+c\mathbf{k}\times\mathbf{B}_0\right)-j(\omega\mathbf{B}_0-\mathbf{k}\times\mathbf{E}_0)\end{aligned} \tag{40}$$

where we have bracketed the product into scalar, vector, and bivector components. Now, each of these components must equal zero separately, for this solution to be valid. Therefore, by inspection, we have the scalar component $\mathbf{k}\cdot\mathbf{E}_0=0$ implying that $\mathbf{k}$ and $\mathbf{E}_0$ are orthogonal, and from the bivector component $\omega\mathbf{B}_0-\mathbf{k}\times\mathbf{E}_0=0$, we see that $\mathbf{B}_0$ is orthogonal to $\mathbf{k}$ and $\mathbf{E}_0$. Hence, we have produced the conventional result that the vectors $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{B}$ are mutually orthogonal for an electromagnetic wave.

Now, because we have removed the abstract imaginary quantity from a representation of the electromagnetic wave, as shown in (38), we can provide a more geometrical depiction in 3-D space, as shown in Fig. 6. Typically, we visualize a propagating electromagnetic wave in the direction $\mathbf{k}$ as consisting of three mutually orthogonal vectors $\mathbf{E},\mathbf{B},\mathbf{k}$. However, once we recognize that the magnetic

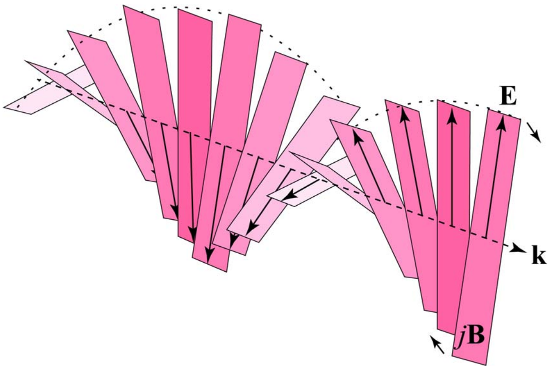

Fig. 6. ***Visual depiction of a circularly polarized plane electromagnetic wave in GA. The electric field is represented by a vector E, lying on a rotating plane representing the magnetic field jB. This figure is adapted from [18].***

field is a bivector quantity represented by an oriented plane, and the vector $\mathbf{B}$ represents the orthogonal vector to this plane, we generate the picture shown in Fig. 6 that shows that the electric field vector $\mathbf{E}$ in fact lies on the plane of the magnetic field $j\mathbf{B}$. This also helps clarify a historical debate on whether to use the magnetic field vector or the electric field vector to characterize the plane of polarization for light [17]. Referring to Fig. 6, we can see clearly that it is more reasonable to use the electric field vector as it also simultaneously represents the plane of the magnetic bivector field.

Consideration of Fig. 6 also leads to an alternative source free solution to Maxwell's equations

$$F = \mathbf{E}_0(1 - \hat{k})e^{j\hat{k}(\omega t - \beta\hat{k}\cdot\mathbf{r})}. \quad (41)$$

The magnetic field now no longer appears explicitly as the magnetic field appears through $\mathbf{E}_0(1 - \hat{k}) = \mathbf{E}_0 - \mathbf{E}_0\hat{k}$, where $j\mathbf{B}_0 = \hat{k}\mathbf{E}_0/c$. The rotating electric field vector at a specified location is now given by the rotation operator $e^{j\hat{k}\omega t}$ that rotates $\mathbf{E}_0$ in the plane $j\hat{k}$ perpendicular to $\hat{k}$. A generalized version of (41) is given in Appendix D for conductive media.

E. Liénard–Wiechert Potentials

We illustrate the efficient representation in GA with a specific example. Given a particle of charge q is moving in a circle of radius a at constant angular velocity ω. Assume that the circle lies in the xy-plane, centered at the origin, and at time $t = 0$, the charge is at $(a, 0)$, on the positive x-axis. Find the Liénard–Wiechert potentials for points on the z-axis. The trajectory of the particle is, therefore, defined as $\mathbf{w}(t_r) = a\cos(wt_r)e_1 + a\sin(wt_r)e_2$.

Using conventional vector analysis, we can find the Liénard–Wiechert potentials from the well-known expressions

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{r'c - \mathbf{r}'\cdot\mathbf{v}}$$
$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2}V(\mathbf{r}, t) \quad (42)$$

where $\mathbf{r}' = \mathbf{r} - \mathbf{w}(t_r)$ and $|\mathbf{r}'| = r' = c(t - t_r)$ where t_r is the retarded time allowing for the propagation speed of light. After significant algebra, we can find

$$V(z, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{s}$$
$$\mathbf{A}(z, t) = \frac{1}{4\pi\epsilon_0}\frac{akq}{sc}(\cos(wt - ks)e_2 - \sin(wt - ks)e_1) \quad (43)$$

where $s = \sqrt{z^2 + a^2}$ and $k = w/c$.

In GA, first the trajectory is described in a simpler form as $\mathbf{w}(t_r) = ae_1e^{e_1e_2wt_r}$, and we also achieve a single unified expression for the potential as

$$\mathbf{A}(z, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{sc}\left(c - ake_2e^{e_1e_2(wt - ks)}\right). \quad (44)$$

Comparing this expression with (43), we can see a significant advantage in compactness.

F. Circuit Analysis and Complex Power

In order to calculate the apparent power in an electrical circuit, we typically use the relation $P = VI^*$. For example, for a series RLC circuit, we have $Z = R + (1/j\omega C) + j\omega L$, where $j = \sqrt{-1}$ and the asterisk is a complex conjugate,[9] and given a source RMS voltage $V = V_{\text{rms}}$, we can find

$$I = \frac{V}{Z} = VZ^{-1} \quad (45)$$

where $Z^{-1} = Z^*/|Z|^2$, with $|Z|^2 = ZZ^*$. This gives the power

$$P = VI^* = \frac{VV^*Z}{|Z|^2} = \frac{V^2Z}{|Z|^2}. \quad (46)$$

However, students typically ask why the power is now $P = VI^*$ rather than simply $P = VI$ as it is for purely resistive circuits.

[9]Notice that just for Section VI-F we have reverted to using j to denote the traditional unit imaginary. Elsewhere throughout this paper j should always be understood to mean the trivector $e_1e_2e_3$.

In terms of GA, we will represent I and V as vectors and the impedance as an operator, so that we have first the vector $V = V_{\rm rms}e_1$. So impedance will be interpreted as a rotation operator for the vector phasors, so that we replace the unit imaginary $\sqrt{-1}$ with e_1e_2. We now have

$$I = \left(\frac{1}{Z}\right)V. \tag{47}$$

This allows us to indeed write $P = VI$ for the power for an alternating current (ac) circuit, where for a source at zero phase $V = V_{\rm rms}e_1$. Checking this relation, we have

$$P = VI = \frac{VZ^*V}{|Z|^2} = \frac{V^2Z}{|Z|^2} = \frac{V^2}{|Z|}\frac{Z}{|Z|} \tag{48}$$

that produces the correct relation.

Hence, GA allows us to write the general relation $P = VI$ that now corresponds with the calculations with simple resistors, as well as instantaneous power calculations with ac circuits. Also, we have $I = Z^{-1}V$, where $V = V_{\rm rms}e_1$ or $V = V_{\rm rms}e^{e_1e_2\theta}$ for a source at a phase angle θ. The formula corresponds closer to the elementary expression $S = V^2/R$ and the phase angle of the power equal to the phase angle of the impedance, which as we know is now interpreted as a rotation operator given by $Z/|Z|$.

VII. RELATIVITY

It is generally believed that special relativistic effects are important only when studying objects moving at speeds close to that of light. This belief leaves many practicing scientists and engineers with the impression that an understanding of relativity is not necessary for their day jobs. This impression is wrong on two counts. First, the principles of relativity are often useful even for slow moving objects, such as deriving the Doppler effect formula [22], and second, modern atomic clocks can easily measure the time dilation effects of satellites in orbit about the Earth—in fact the understanding of this time dilation effect is crucial to the operation of the global positioning system (GPS) [23].

The postulate of SR is simple to state[10] "The laws of physics are identical in all inertial frames ..."[11] [24], but the consequences that include time dilation, length contraction, and mass increase are counterintuitive. Aside from GPS, applications of relativity include: ring laser gyroscopes in aircraft navigation systems, which use the Einstein velocity addition formula [25], nuclear energy ($E = mc^2$) [24], signal processing properties of the light cone [26], and positron emission tomography (PET; relativistic quantum mechanics). From this list, it is clear that the modern engineer working in either signal processing, navigation, or nuclear energy can be expected to have a reasonable working knowledge of relativity theory.

Now, one of the first principles of SR is that each observed particle has its own space and a time coordinate. This is in distinction to Newtonian physics which, while it assigns each particle a spatial coordinate, it assumes a single global time coordinate. In GA, therefore, for each observed particle, we will write a particles' coordinate multivector as

$$X = ct + \mathbf{x} \tag{49}$$

with $\mathbf{x} = x_1e_1 + x_2e_2 + x_3e_3$ representing its position vector and t the time observed on the particles clock. The approach we have adopted of adding a scalar quantity to a spatial vector in order to represent space–time is commonly referred to as the paravector formalism [9], [11], [27], [28]. We then find the space–time interval to be $|X|^2 = X\bar{X} = c^2t^2 - \mathbf{x}^2$ and because it is a scalar, it is an invariant measure of distance under observer transformations that will be defined shortly.

We have from (49) the multivector differential

$$dX = cdt + d\mathbf{x}. \tag{50}$$

For the rest frame of the particle ($d\mathbf{x} = 0$), we have $|dX_0|^2 = c^2d\tau^2$, where we define in this case t to represent the proper time τ of the particle. We have assumed that the speed c is the same in both the rest and the moving frame, as required by Einstein's second postulate regarding the invariance of the speed of light.

To elaborate further the conceptual framework, we see that an observer (such as a terrestrial laboratory) will assign his own unique set of coordinates X to each event that he observes, which in general will not agree with some other observing laboratory while looking at the same events but measuring his own set of coordinates X'. However, the space–time distance will be the same for each event jointly observed, that is, $|X|^2 = |X'|^2$. Each observer will have his own clock that represents his proper time, that will in actuality tick at the same rate as other observers' clocks, nevertheless when an observer's clock is viewed by some other observers, he will consider their clocks to be running slow. That is, if we assume that the

[10] Often two postulates are given for the special theory of relativity; the second is that the speed of light in a vacuum is a constant. This second postulate is a consequence of Maxwell's equations and so it not strictly necessary, though often stated for historical reasons.

[11] Inertial frames (or equivalently observers) are those moving at a constant velocity, i.e., not accelerating.

space–time interval is invariant under the Lorentz transformations defined in (60), then we can equate the rest frame interval to the moving frame interval, giving

$$c^2d\tau^2 = c^2dt^2 - d\mathbf{x}^2 = c^2dt^2 - \mathbf{v}^2dt^2 = c^2dt^2\left(1 - \frac{\mathbf{v}^2}{c^2}\right) \quad (51)$$

with $d\mathbf{x} = \mathbf{v}dt$, and hence, taking the square root, we find the time dilation formula

$$dt = \gamma d\tau \quad (52)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}. \quad (53)$$

This is, in fact, one of the key results from SR that moving clocks will appear to tick at a slower rate than clocks at rest with respect to the observer. This general principle has indeed been extensively verified experimentally, for example, in the increased decay times of fast moving unstable particles in particle accelerators and in decay products from cosmic ray showers, where their extended lifetimes are given by $dt = \gamma d\tau$ where the γ factor approaches infinity as the relative velocity $v \to c$. Time dilation effects have also been confirmed in orbiting satellites such as the GPS satellites, for example, which require a correction of 7.2 μs daily in order to correct for the relativistic time dilation effect. Another example is the Doppler effect of electromagnetic radiation: it turns out the transverse Doppler effect is due solely to relativistic time dilation, where the measured frequency $f = \gamma f_0$, where f_0 is the frequency of the source when at rest [29]. An important corollary of time dilation is that the γ factor depends only on the relative speed and not on the instantaneous acceleration of the object. For example, high-speed particles trapped in circular rings using magnetic fields are subject to very high radial accelerations, however this acceleration does not affect their time dilation as it is only a function of the magnitude of the relative velocity, as shown by (53).

From (50), we can now calculate the proper velocity, differentiating with respect to the proper time, giving the relativistic velocity multivector

$$U = \frac{dX}{d\tau} = \frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} + c\frac{dt}{d\tau} = \gamma c + \gamma\mathbf{v} \quad (54)$$

where we use $dt/d\tau = \gamma$ and define $\mathbf{v} = d\mathbf{x}/dt$. We then find

$$|U|^2 = \left(\frac{1}{1 - \mathbf{v}^2/c^2}\right)(c^2 - \mathbf{v}^2) = c^2. \quad (55)$$

We define the momentum multivector

$$P = mU = \gamma mc + \gamma m\mathbf{v} = \frac{E}{c} + \mathbf{p} \quad (56)$$

with the relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$ and the total energy $E = \gamma mc^2$. This is one case where we break our rule of using lower case for scalars, in order not to deviate from such widespread usage of using E for the scalar valued total energy.

Now, as $|U|^2 = c^2$, then $|P|^2 = m^2c^2$ is an invariant between frames, which gives

$$|P|^2c^2 = E^2 - \mathbf{p}^2c^2 = m^2c^4 \quad (57)$$

the relativistic expression for the conservation of momentum energy. We can, therefore, write the total energy as $E^2 = \mathbf{p}^2c^2 + m^2c^4$, and so for a particle at rest ($\mathbf{p} = 0$), we have the energy $E = mc^2$, Einstein's formula for the energy contained in matter.

The Lorentz force for charges in electromagnetic fields needs to be modified to account for relativistic effects, and in tensor notation written as $K^\mu = (q/mc)F^{\mu\nu}P_\nu$, where $P_\nu = (\gamma mc, -\gamma m\mathbf{v})$ is the 4-momentum. An equivalent expression using geometric algebra is $K = (q/mc)\langle PF\rangle_{01}$. We have selected the scalar and vector components using the angle brackets in order to correspond exactly with the 4-vector for force. The use of P in the force law, remembering that the force $K = dP/d\tau$ means we now have the differential equation

$$\frac{dP}{d\tau} = \left(\frac{q}{mc}\right)\langle PF\rangle_{01}. \quad (58)$$

For the cases where the field $F = \mathbf{E} + jc\mathbf{B}$ is independent of time then we can integrate with respect to time τ to produce an analytic solution $P(t) = e^{ktF^\dagger/2}P(0)e^{ktF/2}$, where $k = q/mc$, for a charge with an initial momentum multivector $P(0)$ in a constant field F. Writing our solution in full, we have

$$P(t) = e^{\frac{kt(\mathbf{E}-jc\mathbf{B})}{2}}P(0)e^{\frac{kt(\mathbf{E}+jc\mathbf{B})}{2}}. \quad (59)$$

A detailed derivation is provided in Appendix E. The position of the particle at any time $X(t)$ is calculated by integrating the momentum with respect to time.

A. The Lorentz Group

The Lorentz transformations describe the transformations for observations between inertial systems in relative motion. We will find that the following operator:

$$L = \mathrm{e}^{\frac{(-\phi\hat{\mathbf{v}}-j\mathbf{w})}{2}} \tag{60}$$

which exponentiates the vector and bivector components of a multivector, will produce the correct transformations for both coordinates and electromagnetic fields, where ϕ is defined through $\tanh\phi = v/c$ where $v = \|\mathbf{v}\|$ and where $\mathbf{v}$ is the relative velocity vector between the observers. The operator $\mathrm{e}^{\phi\hat{\mathbf{v}}/2}$ defines boosts[12] and $\mathrm{e}^{j\mathbf{w}/2}$ allows for measurements in a rotated frame. If we apply this operator to the coordinate multivector X using the transformation

$$X' = LXL^\dagger = \mathrm{e}^{\frac{(-\phi\hat{\mathbf{v}}-j\mathbf{w})}{2}} M \mathrm{e}^{\frac{(-\phi\hat{\mathbf{v}}+j\mathbf{w})}{2}} \tag{61}$$

then we will find that we produce the correct transformation law for coordinates. To see this, we first write a space–time coordinate as $X = ct + \mathbf{x}_\parallel + \mathbf{x}_\perp$, where we split the spatial coordinate into components perpendicular and parallel to the boost direction $\hat{\mathbf{v}}$. For pure boosts, we then find from (61) that

$$\begin{aligned} X' &= \mathrm{e}^{-\hat{\mathbf{v}}\phi/2}\left(ct + \mathbf{x}_\parallel + \mathbf{x}_\perp\right)\mathrm{e}^{-\hat{\mathbf{v}}\phi/2} \\ &= ct\mathrm{e}^{-\hat{\mathbf{v}}\phi} + \mathbf{x}_\parallel \mathrm{e}^{-\hat{\mathbf{v}}\phi} + \mathbf{x}_\perp \\ &= ct(\cosh\phi - \hat{\mathbf{v}}\sinh\phi) + \mathbf{x}_\parallel(\cosh\phi - \hat{\mathbf{v}}\sinh\phi) + \mathbf{x}_\perp. \end{aligned} \tag{62}$$

Now, the expression $\tanh\phi = v/c$ can be rearranged to give $\cosh\phi = \gamma$ and $\sinh\phi = \gamma v/c$. Substituting these relations, we find

$$X' = \gamma\left(ct - \frac{vx_\parallel}{c}\right) + \gamma\left(\mathbf{x}_\parallel - \mathbf{v}t\right) + \mathbf{x}_\perp \tag{63}$$

which thus gives the transformation $\mathbf{x}'_\parallel = \gamma(\mathbf{x}_\parallel - \mathbf{v}t)$, $\mathbf{x}'_\perp = \mathbf{x}_\perp$, and $ct' = \gamma(ct - (vx_\parallel/c))$, the correct Lorentz boost of coordinates, where $x_\parallel = \|\mathbf{x}_\parallel\|$. Normally, due to the complexity of conventional tensor notation, only special cases are calculated, such as a boost along a particular coordinate axis, however because of the simplicity of the notation in GA, we are able to calculate immediately the general case, giving the transformation of both the space and time coordinates in a single equation (63).

The new space–time distance squared will be $X'\bar{X}' = LXL^\dagger(L\bar{X}L^\dagger) = LXL^\dagger\bar{L}^\dagger\bar{X}\bar{L} = X\bar{X}$, because $L\bar{L} = 1$, thus leaving the space–time distance invariant, and so part of the restricted Lorentz group [30]. The transformation defined in (61) was defined to transform coordinates, however the transformation rule also gives the correct transformation for momentum multivectors P as well as the field potential multivector A.

For the transformation of the electromagnetic field multivector $F = \mathbf{E} + jc\mathbf{B}$, we require a slightly different transformation rule

$$F' = LF\bar{L} = \mathrm{e}^{-\phi\hat{\mathbf{v}}-j\mathbf{w}}(\mathbf{E} + jc\mathbf{B})\mathrm{e}^{\phi\hat{\mathbf{v}}+j\mathbf{w}}. \tag{64}$$

The use of SR allows a new solution path when calculating fields of moving charges. We select the rest frame of a moving charge, which implies only electric fields are present, which are given by $\mathbf{E} = q\mathbf{r}/\mathbf{r}^3$, and we then simply transform into a relatively moving frame using the Lorentz transformations to give the electric and magnetic fields, that is, $F = \bar{L}\mathbf{E}L$. However, we also need to allow for the motion of the charge over time, substituting $\mathbf{r} \rightarrow \gamma(\mathbf{r} - \mathbf{v}t)$, where $\mathbf{v}$ is the velocity of the charge. This then gives the full electromagnetic field of a moving charge, without the use of calculus. For example, for a purely electric field $\mathbf{E}$ viewed from a moving observation frame with a relative velocity $\mathbf{v}$, if we split the field into components parallel and perpendicular to the relative velocity vector, given by $\mathbf{E} = \mathbf{E}^\parallel + \mathbf{E}^\perp$, then we find the observed field

$$\begin{aligned} F' &= \mathrm{e}^{-\phi\hat{\mathbf{v}}/2}(\mathbf{E}^\parallel + \mathbf{E}^\perp)\mathrm{e}^{\phi\hat{\mathbf{v}}/2} = \mathbf{E}^\parallel + \mathbf{E}^\perp \mathrm{e}^{\phi\hat{\mathbf{v}}} \\ &= \mathbf{E}^\parallel + \mathbf{E}^\perp(\gamma + \gamma\mathbf{v}/c) = \mathbf{E}^\parallel + \gamma\mathbf{E}^\perp + \gamma\mathbf{E}^\perp \wedge \mathbf{v}/c \\ &= \mathbf{E}^\parallel + \gamma\mathbf{E}^\perp + jc\gamma\mathbf{E}^\perp \times \mathbf{v}/c^2. \end{aligned} \tag{65}$$

We can see, as expected, that the parallel field is unaffected, the perpendicular field is strengthened by the γ factor, and a magnetic component now appears, with $\mathbf{B}' = \gamma\mathbf{E}^\perp \times \mathbf{v}/c^2$.

Typically, in order to represent the transformational properties of the electromagnetic field, a second rank tensor is used, given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & cB_z & -cB_y \\ -E_y & -cB_z & 0 & cB_x \\ -E_z & cB_y & -cB_x & 0 \end{pmatrix} \tag{66}$$

[12] Boosts are rotation-free Lorentz transformations relating the measurements for observers in relative motion.

with the transformed field given by

$$\overline{F}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}F^{\lambda\sigma} \tag{67}$$

where Λ is the 4×4 orthogonal matrix representing the Lorentz transformation.

Hence, the use of $F = \mathbf{E} + jc\mathbf{B}$ being a simple generalization of well-known vector notation, with its transformation properties transparently described by (47), appears preferable to the mathematical overhead of tensors and matrices, shown in (66) and (67).

B. Velocity Addition Rule

If we apply two consecutive parallel boosts, $v_1\hat{\mathbf{v}}$ and $v_2\hat{\mathbf{v}}$ in the direction $\hat{\mathbf{v}}$, then from (61), we have the combined boost operation

$$e^{\frac{\phi_1\hat{\mathbf{v}}}{2}}e^{\frac{\phi_2\hat{\mathbf{v}}}{2}} = e^{\frac{(\phi_1+\phi_2)\hat{\mathbf{v}}}{2}} \tag{68}$$

where $\tanh\phi_1 = v_1/c$ and $\tanh\phi_2 = v_2/c$. We are able to combine the exponents here because parallel vectors commute, as shown by (7). Therefore, we can see that we can write two parallel boosts in terms of a single boost velocity $\tanh(\phi_1 + \phi_2) = v/c$. Hence, we have a new relative velocity between observers of

$$v = c\tanh(\phi_1 + \phi_2) = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1\tanh\phi_2} = \frac{v_1 + v_2}{1 + v_1v_2/c^2} \tag{69}$$

the relativistic velocity addition formula. Hence, with $\mathbf{v}_1$ and then $\mathbf{v}_2$ representing two transitions to a higher relative velocity between observers, we have therefore shown in (69) that the relative velocity between them can never exceed the speed c. That is, if $v_1 \to c$ and $v_2 \to c$, then, from (69), $v \to c$. For low velocities $v \ll c$, we have in the limit for $v_1 \to 0$ and $v_2 \to 0$, then $v \to v_1 + v_2$, the classical velocity addition formula.

The velocity addition formula applies to all objects with mass, whereas for massless electromagnetic radiation, the speed will always be measured as c for all observers in accordance with the principles of the special theory of relativity.

C. Larmor Precession

The electron e^- is the fundamental particle responsible for the formation of electric currents, but besides its unit electric charge $1.602176565(35) \times 10^{-19}$ C the particle has a magnetic moment of $\mu = -9.284764 \times 10^{-24}$ Joule/Tesla.

For the case of an electron orbiting a purely electric positive charge, as in the classical model of an electron orbiting a proton for the hydrogen atom, we might conclude that because the electron is moving through a purely electric field, then the magnetic moment of the electron will not come into play. This would, in fact, be a wrong conclusion, however, as can immediately be seen through an application of the principle of relativity. In this case, we can move to an observer frame that is sitting on the electron itself that we now consider to be at rest, and so observe the positive charge now orbiting the electron. Now, as we know, a moving electric field produces a magnetic field $\mathbf{B} = \gamma\mathbf{E} \wedge \mathbf{v}/c^2$ and hence there will now be a torque $\tau = \mu \times \mathbf{B}$ that will cause the electron to precess. This effect is called Larmor precession observed to cause the splitting of atomic spectral lines, although, when calculated in this manner, we in fact produce an effect twice that observed in atomic spectra. This is because we have not yet allowed for a second relativistic effect called Thomas precession that is discussed in Section VII-D.

D. Length Contraction

For a rigid rod moving directly toward us, we would measure a space–time coordinate of the near end $X_1 = ct_1 + \mathbf{x}_1$ and for the far end $X_2 = ct_2 + \mathbf{x}_2$. Hence, we find the space–time difference $L = X_2 - X_1 = \mathbf{x}_2 - \mathbf{x}_1$.

Using the Lorentz transformation, we find $X' = \gamma(ct - (vx_\parallel/c)) + \gamma(\mathbf{x}_\parallel - \mathbf{v}t) + \mathbf{x}_\perp$, where in this case we have $\mathbf{x}_\perp = 0$ and $\mathbf{v}$ is the velocity of the rod

$$\begin{aligned} X_1' &= \gamma\left(ct_1 - \frac{vx_1}{c}\right) + \gamma(\mathbf{x}_1 - \mathbf{v}t_1) \\ X_2' &= \gamma\left(ct_2 - \frac{vx_2}{c}\right) + \gamma(\mathbf{x}_2 - \mathbf{v}t_2). \end{aligned} \tag{70}$$

Now, in order to measure meaningfully the length of a moving stick, we need to measure each end at the same time, that is, $t_1 = t_2$, and so we find

$$\begin{aligned} L' &= X_2' - X_1' = \gamma c(t_2 - t_1) + \gamma((\mathbf{x}_2 - \mathbf{x}_1) - \mathbf{v}(t_2 - t_1)) \\ &= \gamma(\mathbf{x}_2 - \mathbf{x}_1) = \gamma L. \end{aligned} \tag{71}$$

This result implies that an observer sees length contraction on moving objects. That is, L' is interpreted as the length you would be expected to measure if you slowed the rod down and measured it in your own rest frame. Hence, not only time but also length (in the direction of motion) is shrunk by the γ factor. Lengths perpendicular to the direction of motion will be unaffected. Once again, at first glance, it might appear that Lorentz contraction has little to do with ordinary electromagnetic theory. However, for the case of two parallel wires carrying a current in the same direction, for example, from a special relativistic perspective, the moving electron in one wire sees the positive charges in the other wire Lorentz contracted closer together and so of higher density than the electrons, and will experience an effective electric attractive force. Coincidentally, this turns out to be the same force calculated with the conventional approach assuming that the wires all generate magnetic fields that interact.

Now, returning to our application of an electron orbiting a proton, and assuming a circular orbit, we can see that the orbit is shrunk in the direction of motion due to Lorentz contraction and so the electron will appear to an outside observer to be turning a sharper angle than necessary to orbit the proton. This special relativistic precession effect is called the Thomas precession [31], which now allows accurate prediction of spectral lines. Refer to Appendix F for a detailed calculation of the Thomas precession from the Lorentz contraction.

Now, because the Thomas precession is a geometrical effect, it applies uniformly to all orbiting particles and so is applicable to both atomic orbitals and satellite orbits such as GPS satellites.

The phenomenon of Thomas precession can be derived more formally using the boost operators from (44), where we have a sequence of radial boosts that keep the particle in circular orbit. That is, for this sequence of nonparallel radial boosts, we have

$$\begin{aligned} L &= \mathrm{e}^{-\phi_2\hat{\mathbf{v}}_2/2}\mathrm{e}^{-\phi_1\hat{\mathbf{v}}_1/2} \\ &= \cosh\frac{\phi_2}{2}\cosh\frac{\phi_1}{2} + \hat{\mathbf{v}}_1\cdot\hat{\mathbf{v}}_2\sinh\frac{\phi_2}{2}\sinh\frac{\phi_1}{2} \\ &\quad - \hat{\mathbf{v}}_1\cosh\frac{\phi_2}{2}\sinh\frac{\phi_1}{2} - \hat{\mathbf{v}}_2\cosh\frac{\phi_1}{2}\sinh\frac{\phi_2}{2} \\ &\quad + \hat{\mathbf{v}}_1\wedge\hat{\mathbf{v}}_2\sinh\frac{\phi_2}{2}\sinh\frac{\phi_1}{2} \end{aligned} \tag{72}$$

which consists of scalar, vector, and bivector components. We can see, therefore, that we cannot write this as a single equivalent boost $\mathrm{e}^{\phi_3\hat{\mathbf{v}}_3/2} = \cosh(\phi_3/2) + \hat{\mathbf{v}}_3\sinh(\phi_3/2)$ due to the presence of the bivector term $\hat{\mathbf{v}}_1\wedge\hat{\mathbf{v}}_2\sinh(\phi_2/2)\sinh(\phi_1/2)$. This bivector term represents a rotation and leads to the Thomas rotation, as derived earlier.

E. Application: Doppler Shift

The Doppler shift of light refers to the change of frequency caused by the relative velocity between the source and the observer. In the rest frame of the source, we can describe a single wavelength λ of emitted light using (49), setting up the e_1-axis along the line of sight as

$$X = cT + \lambda e_1 = \lambda + \lambda e_1 \tag{73}$$

where $T = \lambda/c$ is the period of the wave, which gives $|X|^2 = 0$ as required for a photon. We can describe an observer in relative motion with a boost in the $\hat{\mathbf{v}} = e_1$ direction using $\tanh\phi = |\mathbf{v}|/c$, and we find from (61)

$$X' = \gamma\lambda\left(1 - \frac{v}{c}\right) + \gamma\lambda\left(1 - \frac{v}{c}\right)e_1. \tag{74}$$

So using the space (or alternatively time) component, we find $\lambda' = \lambda\gamma(1 - (v/c))$, and using $c = f\lambda$, we find the relativistic Doppler shift formula

$$\frac{f\prime}{f} = \frac{1}{\gamma\left(1 - \frac{v}{c}\right)} = \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}}. \tag{75}$$

VIII. OUTLOOK

The development of GA is now expanding rapidly in many areas of science with engineering applications including terahertz spectroscopy, which allows efficient processing of time domain signals [33], neural networks [34], nonsinusoidal electrical power [35], [36], anisotropic materials and metamaterials, providing a more general description for metamaterials that may allow new innovations [19], [37], quantum computing allowing a more intuitive understanding of quantum algorithms such as the Grover search algorithm [38], support vector machines [39], quantum game theory [40]–[42], perfect electromagnetic conductors [43], robotics using conformal geometric algebra [44], [45], and computer vision [46].

Many problems in SR can often be reduced to a planar framework that also significantly simplifies analysis [32].

It is important to note that two distinct transformation rules have now been defined, one for fields and another one for coordinates. For more advanced treatments, it is often desirable to have a single universal transformation law that transforms all quantities uniformly. One popular approach is to increase the size of our algebra from $C\ell(\Re^3)$ to $C\ell(\Re^{1,3})$ where we now raise time to the status of a fourth dimension. In this case, we can give the unit vector for time as e_0, say, we require $e_0^2 = -1$ as opposed to plus one for space vectors. Also, with the addition of one more dimension, the space doubles in size from 8 to 16 basis elements. Provided this extra complexity is acceptable, then we can adopt this space for full relativistic analysis [13], [27], [28].

Clifford's geometric algebra appears to be an idea whose time has come with a recent article in *Nature Physics* [47], suggesting that because of its superior geometric intuition one day it will be taught in high schools in place of Heaviside–Gibbs vector analysis.

IX. CONCLUSION

Due to the difficulties of representing Maxwell's equations using quaternions, Heaviside rejected Hamilton's algebraic system and developed a system of vectors using the dot and cross products, which is the conventional system used today.

Through identifying Hamilton's quaternions with the three bivectors e_1e_2, e_3e_1, e_2e_3, it is now possible to resolve the dispute between the champions of Hamilton's quaternions and the supporters of Heaviside's vectors, through realizing that Heaviside vectors represent the three translational freedoms of physical space e_1, e_2, e_3, and Hamilton's quaternions represent the three rotational freedoms of space e_1e_2, e_3e_1, e_2e_3, as shown in Fig. 4. In addition, the unit imaginary that is used to produce complex numbers can be superseded with the trivector $e_1e_2e_3$ of 3-D space. Hence, these competing systems are now unified within Clifford's system.

Regarding the role of complex numbers and quaternions, Baez has commented: "The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings" [48, p. 145]. With Clifford's system, this distinction dissolves with the complex numbers and quaternions now both placed on an equal footing with Cartesian vectors within a real algebraic system.

With the introduction of bivectors and trivectors in GA, in addition to conventional vectors, we now have a more appropriate representation for the magnetic field as a bivector field, as well as allowing a single electromagnetic field variable $\mathbf{E} + jc\mathbf{B}$. We can also represent planes directly as bivectors, rather than through the use of a perpendicular vector. The general notational simplification found using GA is illustrated in Table 3. In engineering, we are always taught to check a formula dimensionally; moreover, GA provides an additional structural check, specifically that the calculated quantity has to be the correct algebraic order, whether a scalar, vector, bivector, or trivector quantity.

Typically engineers look for the simplest formula to produce results that are accurate enough for the task at hand. Hence, the value of simplifying the representation of Maxwell's equations into a single equation and using notation that naturally embodies the nature of the quantities modeled produces compact representation and increased intuition for the various physical relationships.

In conclusion, with the Clifford algebra of three dimensions $C\ell(\Re^3)$, we find an elegant algebraic model of physical 3-D space and time that completes Gibbs' vector formalism, removes the distinction between polar and axial vectors, simplifies many formulas, allows a relativistic treatment, and provides additional geometric insight to many problems. ■

APPENDIX A

MAXWELL'S ORIGINAL ELECTROMAGNETIC EQUATIONS

Maxwell in his treatise of 1865 [49] collected together the electromagnetic equations as follows:

$$\begin{aligned}
0 &= e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} \\
\mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz} \quad \mu\beta = \frac{dF}{dz} - \frac{dH}{dx} \quad \mu\gamma = \frac{dG}{dx} - \frac{dF}{dy} \\
P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\phi}{dx} \\
Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\phi}{dy} \\
R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\phi}{dz} \\
\frac{d\gamma}{dy} - \frac{d\beta}{dz} &= 4\pi\left(p + \frac{df}{dt}\right) \\
\frac{d\alpha}{dz} - \frac{d\gamma}{dx} &= 4\pi\left(q + \frac{dg}{dt}\right) \\
\frac{d\beta}{dx} - \frac{d\alpha}{dy} &= 4\pi\left(r + \frac{dh}{dt}\right) \\
0 &= \frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} \\
P &= kf \quad Q = kg \quad R = kH
\end{aligned} \tag{76}$$

where the correspondence with modern vector notation is

$$\begin{aligned}
&\mathbf{E} = (P, Q, R) \quad \mathbf{D} = (f, g, h) \quad \mathbf{H} = (\alpha, \beta, \gamma) \\
&\mathbf{A} = (F, G, H) \quad \mathbf{J} = (p, q, r)
\end{aligned} \tag{77}$$

and $\rho = -e$, $k = 1/\epsilon$, and ϕ is the electric potential. Using this vector notation, we can now write Maxwell's equations as

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\mu\mathbf{H} &= \nabla \times \mathbf{A} \\
\mathbf{E} &= \mu\mathbf{v} \times \mathbf{H} - \frac{dA}{dt} - \nabla\phi \\
\nabla \times \mathbf{H} - 4\pi\frac{d\mathbf{D}}{dt} &= 4\pi\mathbf{J} \\
\nabla \cdot J + \rho &= 0 \\
\mathbf{D} &= \epsilon\mathbf{E}.
\end{aligned} \tag{78}$$

We notice that the Lorentz force law is included as part of the third equation. However, Maxwell states that this term disappears if there is no motion of the conductor, and hence we can ignore this for the microscopic case. The last equation is normally kept separate from the main set of equations as constitutive relations, along with $\mathbf{B} = \mu\mathbf{H}$, and also the fifth equation, being the continuity equation for charge, can be recovered from the fourth equation through taking the divergence, using the fact that the divergence of a curl is zero. With the second equation, after taking the divergence, we find $\nabla\cdot(\mu\mathbf{H}) = \nabla\cdot\mathbf{B} = 0$. Also, with the third equation, if we take the curl, we find $\nabla\times\mathbf{E} = -\mu(dH/dt)$, where the curl of a gradient is zero. We now have the four equations

$$\begin{aligned}\nabla\cdot\mathbf{D} &= \rho\\ \nabla\cdot\mathbf{H} &= 0\\ \nabla\times\mathbf{D} &= -\mu\epsilon\frac{dH}{dt}\\ \nabla\times\mathbf{H} - 4\pi\frac{d\mathbf{D}}{dt} &= 4\pi\mathbf{J}\end{aligned} \tag{79}$$

which are the conventional four equations for linear media that can be compared with (1). Maxwell has also defined the potentials through $\mathbf{E} = -(dA/dt) - \nabla\phi$ and $\mu\mathbf{H} = \mathbf{B} = \nabla\times\mathbf{A}$. It is interesting to note that Maxwell used the full time derivative, whereas later on, he switched to partial derivatives after analyzing the homopolar generator.

APPENDIX B

MORE GENERAL COORDINATE SYSTEMS

For simplicity, this paper utilizes an orthonormal Cartesian system, however, different coordinate systems can be defined.

Selecting a nonorthogonal basis, such as $\mathbf{v}_1$ and $\mathbf{v}_2$, we produce a vector $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$, where $\mathbf{v}_1\mathbf{v}_1 = \|\mathbf{v}_1\|^2$ is not necessarily equal to one and $\mathbf{v}_1$ is not necessarily orthogonal to $\mathbf{v}_2$. We then produce

$$\begin{aligned}\mathbf{v}^2 &= (a\mathbf{v}_1 + b\mathbf{v}_2)^2\\ &= a^2\mathbf{v}_1^2 + b^2\mathbf{v}_2^2 + ab(\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2\mathbf{v}_1).\end{aligned} \tag{80}$$

Now, for two vectors in GA, we have the general result that $\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2\mathbf{v}_1 = 2\mathbf{v}_1\cdot\mathbf{v}_2 = 2|\mathbf{v}_1||\mathbf{v}_2|\cos\theta$, where θ is the included angle. Hence

$$\mathbf{v}^2 = a^2\mathbf{v}_1^2 + b^2\mathbf{v}_2^2 + 2ab|\mathbf{v}_1||\mathbf{v}_2|\cos\theta \tag{81}$$

which is the cos rule for summing two vectors $a\mathbf{v}_1$ and $b\mathbf{v}_2$, and so this produces the correct vector length.

If we wish to employ a covariant basis to a nonorthogonal contravariant basis $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$, then we can produce the reciprocal basis

$$\mathbf{a}^1 = \frac{\mathbf{a}_2\wedge\mathbf{a}_3}{\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3}\quad \mathbf{a}^2 = \frac{\mathbf{a}_3\wedge\mathbf{a}_1}{\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3}\quad \mathbf{a}^3 = \frac{\mathbf{a}_1\wedge\mathbf{a}_2}{\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3}. \tag{82}$$

The correspondence with conventional vector analysis is $\mathbf{a}_1\wedge\mathbf{a}_2 = j\mathbf{a}_1\times\mathbf{a}_2$ and $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3 = j\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{a}_3)$, where j is the trivector. The triple wedge product $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3$ is a pure trivector and so is commuting with all other quantities. In terms of the geometric product, we can write $\mathbf{a}_1\wedge\mathbf{a}_2 = (1/2)(\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}_1)$ and $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3 = (1/2)\times(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1)$. We can, therefore, write the reciprocal basis in terms of the geometric product as

$$\begin{aligned}\mathbf{a}^1 &= \frac{\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}\quad \mathbf{a}^2 = \frac{\mathbf{a}_3\mathbf{a}_1 - \mathbf{a}_1\mathbf{a}_3}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}\\ \mathbf{a}^3 &= \frac{\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}_1}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}.\end{aligned} \tag{83}$$

In this form, it is quick to verify that $\mathbf{a}^1\cdot\mathbf{a}_1 = \mathbf{a}^2\cdot\mathbf{a}_2 = \mathbf{a}^3\cdot\mathbf{a}_3 = 1$ and $\mathbf{a}_p\cdot\mathbf{a}^q = \mathbf{a}^p\cdot\mathbf{a}_q = 0$ for $p\neq q$. For example

$$\begin{aligned}\mathbf{a}_1\cdot\mathbf{a}^2 &= \frac{1}{2}(\mathbf{a}_1\mathbf{a}^2 + \mathbf{a}^2\mathbf{a}_1)\\ &= \frac{1}{2}\frac{\mathbf{a}_1\mathbf{a}_3\mathbf{a}_1 - \mathbf{a}_1\mathbf{a}_1\mathbf{a}_3}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1} + \frac{1}{2}\frac{\mathbf{a}_3\mathbf{a}_1\mathbf{a}_1 - \mathbf{a}_1\mathbf{a}_3\mathbf{a}_1}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}\\ &= \frac{1}{2}\frac{\mathbf{a}_1\mathbf{a}_3\mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_3 - \mathbf{a}_1\mathbf{a}_3\mathbf{a}_1}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1} = 0\end{aligned} \tag{84}$$

and

$$\begin{aligned}\mathbf{a}_2\cdot\mathbf{a}^2 &= \frac{1}{2}(\mathbf{a}_2\mathbf{a}^2 + \mathbf{a}^2\mathbf{a}_2)\\ &= \frac{1}{2}\frac{\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1 - \mathbf{a}_2\mathbf{a}_1\mathbf{a}_3}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1} + \frac{1}{2}\frac{\mathbf{a}_3\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}_1\mathbf{a}_3\mathbf{a}_2}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}\\ &= \frac{1}{2}\frac{(\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1 - \mathbf{a}_1\mathbf{a}_3\mathbf{a}_2) + (\mathbf{a}_3\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}_1\mathbf{a}_3)}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}.\end{aligned} \tag{85}$$

Now, $\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1 - \mathbf{a}_1\mathbf{a}_3\mathbf{a}_2 = 2\mathbf{a}_2\wedge\mathbf{a}_3\wedge\mathbf{a}_1 = 2\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3$ and $\mathbf{a}_3\mathbf{a}_1\mathbf{a}_2 - \mathbf{a}_2\mathbf{a}_1\mathbf{a}_3 = 2\mathbf{a}_3\wedge\mathbf{a}_1\wedge\mathbf{a}_2 = 2\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3$ using the fact that the wedge product is antisymmetric and associative. That is, $\mathbf{a}_1\wedge(\mathbf{a}_2\wedge\mathbf{a}_3) = (\mathbf{a}_1\wedge\mathbf{a}_2)\wedge\mathbf{a}_3$ and $\mathbf{a}_1\wedge\mathbf{a}_2 = -\mathbf{a}_2\wedge\mathbf{a}_1$. Hence

$$\begin{aligned}\mathbf{a}_2\cdot\mathbf{a}^2 &= \frac{1}{2}\frac{(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1) + (\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1)}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}\\ &= \frac{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1}{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 - \mathbf{a}_3\mathbf{a}_2\mathbf{a}_1} = 1\end{aligned} \tag{86}$$

as required. Hence, even if neither the contravariant nor the covariant basis is normed to one, their products $\mathbf{a}^1\mathbf{a}_1 = \mathbf{a}^2\mathbf{a}_2 = \mathbf{a}^3\mathbf{a}_3$ will equal one. Hence, we can use either (82) or (83) to define the reciprocal basis. For the special case of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ being the orthonormal basis e_1, e_1, e_3, we find using (83), the covariant basis vector for example

$$e^1 = \frac{e_2e_3 - e_3e_2}{e_1e_2e_3 - e_3e_2e_1} = \frac{2e_2e_3}{2e_1e_2e_3} = \frac{je_1}{j} = e_1 \qquad (87)$$

as expected. Therefore, writing a vector with contravariant components $X^\mu = x^\mu \mathbf{a}_\mu$ and also with covariant components using the reciprocal basis $X_\mu = x_\mu \mathbf{a}^\mu$, we find the invariant quantity

$$X \cdot X = X^\mu X_\mu = x^\mu \mathbf{a}_\mu x_\mu \mathbf{a}^\mu = x^\mu x_\mu \qquad (88)$$

using $\mathbf{a}_\mu \cdot \mathbf{a}^\mu = 1$.

APPENDIX C

MAGNETIC MONOPOLES IN GA

Inspecting the detailed form of Maxwell's equations in (19), we can see that the multivector variables are not fully populated. For example, the source terms on the right-hand side consist of just the scalar and vector components, and we might, therefore, attempt to complete the multivector by adding bivector and trivector sources as follows:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)(\mathbf{E} + jc\mathbf{B}) = \rho - c\mu_0\mathbf{J} - j\mu_0\mathbf{J^m} + jc\mu_0\rho^m. \qquad (89)$$

Maxwell's equations are now modified to $\nabla \cdot B = \rho^m$ and $\nabla \times \mathbf{E} + (\partial\mathbf{B}/\partial t) = -\mathbf{J}^m$. This, in fact, is the form of Maxwell's equation if we include the presence of monopoles, where ρ^m represents magnetic charge and $\mathbf{J}^m$ is the current of magnetic charge. No free monopoles have yet been found, but we have illustrated how they can be naturally added to Maxwell's equations using GA and perhaps how they are conspicuously absent.

Regarding fully populating the electromagnetic field to $F = l + \mathbf{E} + jc\mathbf{B} + jc$, we have already observed that l represents the Lorenz gauge, which is set to zero for fundamental reasons of conservation of energy and causality, however the trivector term may also potentially be investigated.

APPENDIX D

ELECTROMAGNETIC WAVES IN CONDUCTIVE MEDIA

We have a solution to the source free Maxwell equation in conductive media as

$$\begin{aligned} F &= \mathbf{E}_0\left(1 + \frac{jc}{\omega}\Gamma\right)e^{j\hat{\mathbf{k}}\omega t - \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} \\ &= \mathbf{E}_0\left(1 + \left(\frac{\alpha c}{\omega}\right)j - \left(\frac{\beta c}{\omega}\right)\hat{\mathbf{k}}\right)e^{j\hat{\mathbf{k}}\left(\omega t - \beta\hat{\mathbf{k}}\cdot\mathbf{r}\right)}e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}} \end{aligned} \qquad (90)$$

where $\Gamma = \sqrt{j\hat{\mathbf{k}}\omega\mu(\sigma + j\hat{\mathbf{k}}\omega\epsilon)} = \alpha + j\hat{\mathbf{k}}\beta$. We have $\beta = (\omega/c)\sqrt{(1/2)(1 + \sqrt{1 + (\sigma/\omega\epsilon)^2})}$ and $\alpha = \sigma\mu\omega/(2\beta)$

$$\begin{aligned} \partial F &= \left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)\mathbf{E}_0\left(\frac{1\pm}{\Gamma cj\omega}\right)e^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} \\ &\left(-\frac{\omega}{c}j\hat{\mathbf{k}} \pm \Gamma\hat{\mathbf{k}}\right)\mathbf{E}_0\left(1 \pm \frac{\Gamma c}{j\omega}\right)ve^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} \\ &\mathbf{E}_0\frac{\omega}{c}\left(j\hat{\mathbf{k}} \mp \frac{c}{\omega}\Gamma\hat{\mathbf{k}}\right)\left(1 \pm \frac{\Gamma c}{j\omega}\right)e^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} \\ &\mathbf{E}_0\frac{\omega j\hat{\mathbf{k}}}{c}\left(1 \mp \frac{\Gamma c}{j\omega}\right)\left(1 \pm \frac{\Gamma c}{j\omega}\right)e^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} \\ &\mathbf{E}_0\frac{\omega j\hat{\mathbf{k}}}{c}\left(1 + \frac{\Gamma^2c^2}{\omega^2}\right)e^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}}. \end{aligned} \qquad (91)$$

Now

$$\begin{aligned} 1 + \frac{\Gamma^2c^2}{\omega^2} &= 1 + \left(j\hat{\mathbf{k}}\omega\mu(\sigma + j\hat{\mathbf{k}}\omega\epsilon)\right)\frac{c^2}{\omega^2} \\ &= 1 + \left(j\hat{\mathbf{k}}\omega\mu\sigma - \omega^2\mu\epsilon\right)\frac{c^2}{\omega^2} = j\hat{\mathbf{k}}\mu\sigma\frac{c^2}{\omega}. \end{aligned} \qquad (92)$$

Therefore

$$\begin{aligned} \partial F &= \mathbf{E}_0\frac{\omega j\hat{\mathbf{k}}}{c}\left(j\hat{\mathbf{k}}\mu\sigma\frac{c^2}{\omega}\right)e_1p^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} = -\mathbf{E}_0(\mu\sigma c)e^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} \\ &= -\mu\sigma c\mathbf{E}_0e^{j\hat{\mathbf{k}}\omega t \pm \Gamma\hat{\mathbf{k}}\cdot\mathbf{r}} = -\mu c\sigma\mathbf{E}. \end{aligned} \qquad (93)$$

Therefore, we have a generated current $J = \sigma\mathbf{E}$ as assumed in conductive medium with conductivity σ. So we now satisfy the Maxwell equation

$$\partial F = -\mu c\sigma\mathbf{E} \qquad (94)$$

where $F = \mathbf{E} + jc\mathbf{B}$.

For the lossless case with $\sigma = 0$, we find

$$F = \mathbf{E}_0(1 - \hat{\mathbf{k}})e^{j\hat{\mathbf{k}}(\omega t - \beta\hat{\mathbf{k}}\cdot\mathbf{r})} \tag{95}$$

where $\beta = w/c$. Hence, this solution implies that an electromagnetic wave consists of solely a propagating electric field vector with the magnetic field only arising with respect to massive observers. This idea can be supported by the well-known relation for the magnetic field generated by a moving electric field vector $\mathbf{B} = -(1/c^2)\mathbf{v} \times \mathbf{E}$. For $\mathbf{v}$ having a speed of c, we find $j\mathbf{B} = \hat{v}\mathbf{E}/c$ as we assumed for the electromagnetic wave.

APPENDIX E

PROOF OF SOLUTION TO THE LORENTZ FORCE EQUATION

From (58), we have the differential equation

$$\frac{dP}{d\tau} = \left(\frac{q}{mc}\right)\langle PF\rangle_{01} = \left(\frac{q}{2mc}\right)\left(PF + (PF)^{\dagger}\right). \tag{96}$$

The reversion operation, represented by the tilde, reverses the sign of the bivector and trivector components and so can be utilized to remove these components as shown. Remembering that $(PF)^{\dagger} = F^{\dagger}\bar{P}$, we can, therefore, write (96) as

$$\frac{dP}{d\tau} = \left(\frac{q}{2mc}\right)\left(PF + F^{\dagger}P^{\dagger}\right). \tag{97}$$

Now, we have the proposed solution

$$P(\tau) = e^{k\tau F^{\dagger}}P(0)e^{k\tau F} \tag{98}$$

where $k = q/2mc$ for a charge with an initial momentum multivector $P(0) = E(0)/c + \mathbf{p}(0)$, in a constant field $F = \mathbf{E} + jc\mathbf{B}$. We note first that $P(\tau) = P(\tau)^{\dagger}$ because the momentum multivector consists of just scalar and vector components. Using our solution for $P(\tau)$, we find

$$\frac{dP}{d\tau} = e^{k\tau F^{\dagger}}P(0)e^{k\tau F}kF + kF^{\dagger}e^{k\tau F^{\dagger}}P(0)e^{k\tau F} = k(PF + F^{\dagger}P) \tag{99}$$

using the product rule for differentiation and respecting noncommutivity, thus satisfying (97), as required.

APPENDIX F

THOMAS PRECESSION

We now calculate the Thomas precession for an orbiting object in a circular orbit. If we align the x-axis with the instantaneous direction of motion of the satellite, then for a small translation dx, the satellite will need to deflect some distance dy toward the center in order to stay in orbit.

So we find in the rest frame of the satellite that

$$\tan d\theta = \frac{dy}{dx}. \tag{100}$$

However, when the orbiting object is viewed from the center, then the length is contracted in the direction of motion and so we have $\tan d\theta' = dy/(dx/\gamma) = \gamma dy/dx = \gamma \tan d\theta$. For infinitesimal angles, we have $\tan d\theta = d\theta$ and so we have the relation $d\theta' = \gamma d\theta$. Hence, for a complete orbit of $\theta = 2\pi$, the satellite will be observed to rotate an angle $\gamma 2\pi$. Hence, the excess rotation will be

$$\Omega = \gamma 2\pi - 2\pi = 2\pi(\gamma - 1) \tag{101}$$

which is the Thomas precession. If we expand this into a power series we find to lowest order $\Omega = 2\pi v^2/c^2$. When relativistic correction of the Thomas precession is added to the Larmor precession calculated earlier, then we find a near-exact correspondence with the observed spectral emission from atoms.

APPENDIX G

SIMPLE ILLUSTRATIVE EXAMPLES OF USING GA

1) An $\mathbf{H}$ field travels in the $-e_3$ direction in free space with a constant phase shift of 30.0 rad/m and an amplitude of $1/3\pi$ A/m. If the field has the direction $-e_2$ when $t = 0$ and $z = 0$, then write suitable expressions for $\mathbf{E}$ and $\mathbf{H}$.We have the wave propagation direction $\hat{\mathbf{k}} = -e_3$ and in lossless free space $\omega = \beta c_0 = 9 \times 10^9$ rad/s and $120Z_0 \approx \pi\Omega$. For linear polarization

$$\begin{aligned}\mathbf{H}(\mathbf{r}, t) &= \mathbf{H}_0 \cos(\omega t - \beta\mathbf{k}\cdot\mathbf{r})\\ &= -\frac{1}{3\pi}e_2\cos(9\times10^9 t + 30z)\ \text{A/m}. \end{aligned} \tag{102}$$

The electromagnetic field is thus

$$\begin{aligned}F(\mathbf{r}, t) &= (\hat{\mathbf{k}} + 1)jZ_0\mathbf{H}(\mathbf{r}, t)\\ &= 40(e_1 - je_2)\cos(9\times10^9 t + 30z)\ \text{V/m}\end{aligned}$$

from which the electric field is easily extracted

$$\mathbf{E}(\mathbf{r},t) = \langle F(\mathbf{r},t)\rangle_1 = 40e_1\cos(9\times 10^9 t + 30z)\ \text{V/m}. \quad (103)$$

2) A wire of length 2.5 m located at $z = 0$, $x = 4$ m carries a current of $I = 12.0$ A in the $-e_2$ direction. Find the uniform $\mathbf{B}$ field in the region if the force on the conductor is 1.20 N in the direction $(-e_1 + e_3)/\sqrt{2}$. Given that the wire is given as the vector $\mathbf{w} = 2.5e_2$ m, then we have the force equation

$$\mathbf{F} = -I\mathbf{w}\cdot(j\mathbf{B}). \quad (104)$$

This can be inverted to give

$$I\mathbf{w}(j\mathbf{B}) = I\mathbf{w}\wedge(j\mathbf{B}) - \mathbf{F} \quad (105)$$

giving the magnetic field explicitly as

$$j\mathbf{B} = \mathbf{w}^{-1}(\mathbf{w}\wedge(j\mathbf{B})) - \mathbf{w}^{-1}\mathbf{F}/I = j\hat{\mathbf{w}}B^{\parallel} - \mathbf{w}^{-1}\mathbf{F}/I \quad (106)$$

that splits into a field parallel (represented by the scalar $B^{\parallel}$) and perpendicular to the wire. Note that first the exterior product of a vector and bivector is necessarily a volume or pseudoscalar so we can replace it with j and that $\mathbf{w}^{-1}$ is parallel to $\mathbf{w}$ given by the unit vector $j\hat{\mathbf{w}}$. The parallel component does not contribute to the force so is, therefore, not determinable. Hence, we can write the $\mathbf{B}$ field vector as

$$\mathbf{B} = j\mathbf{w}^{-1}\mathbf{F}/I + k\hat{\mathbf{w}} = j\mathbf{w}\mathbf{F}/(I\mathbf{w}^2) + k\hat{\mathbf{w}}$$

that is now written only in terms of the geometric product where $k = B^{\parallel}$ is a parameter giving the possible magnetic field parallel to the wire.Note also that the dot product between the vector and the bivector is equivalent to the negative of the cross product between vectors and results in a vector and that the geometric product is the sum of the inner and generalized outer products.Given that $\mathbf{w}^{-1} = 0.4e_2\ \text{m}^{-1}$, then

$$j\mathbf{B} = 0.4e_2(2.5e_2\wedge(j\mathbf{B})) - 0.4e_2 1.2(-e_1+e_3)/\sqrt{2}/12.0 = jB_y e_2 - 4\times 10^{-2} j(e_1+e_3)/\sqrt{2}\ \text{T} \quad (107)$$

where B_y is any scalar giving the field parallel to the wire.

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