ABSTRACT EROSION BY A LOAD TRAVERSING A ROUTED NETWORK ON A DIRECTED GRAPH WITH APPLICATIONS TO PROCESSES AND GAMES

Definition. A **directed graph** is a quadruple $(X, X', \perp, \uparrow)$ such that $\perp, \uparrow \colon X \to X'$ are maps that give the **base** and **head** of an **edge** in X where X' is the set of **nodes**. Note that the map $\perp \times \uparrow \colon X \to X' \times X'$ given by $(\perp \times \uparrow)(x) = (\perp (x), \uparrow (x))$ is not necessarily injective—there may be more than one edge with base $x' \in X'$ and head $y' \in X'$.

Definition. A cancellative monoid C is a **halfgroup** if its group of units is trivial, in symbols $C^\times = \{1\}$. A **homomorphism** of halfgroups is a monoid homomorphism.

Exercise. Prove that a nontrivial halfgroup has infinitely many elements.

The shorthand Halfgroups is used as follows: by "$f \colon X \to$ Halfgroups is a map" is meant there is some implicit set Y such that each element of Y is a halfgroup, and $f \colon X \to Y$ is a map.

WARNING: In general, the quotient of a halfgroup by a monoid congruence is not a halfgroup, as it may posses nontrivial invertible elements.

Exercise. Let I be an index set and $(C_i)_{i \in I}$ an indexed collection of halfgroups. Prove that the direct product $\bigtimes_{i \in I} C_i$ and direct sum $\bigoplus_{i \in I} C_i$ are halfgroups.

Definition. Let C and D be halfgroups. A map $a \colon C \to D$ is an **accumulator** if $a(1) = 1$ and $a(cd) \in a(c)D$ for all $c, d \in C$.

The shorthand Accumulators is used as follows: by "$f \colon X \to$ Accumulators is a map" is meant there are implicit sets Y and Z such that each each element of Y is a halfgroup, each element of Z is a triple (a, C, D) where $C, D \in Y$, $a \colon C \to D$ is an accumulator, and $f \colon X \to Z$ is a map.

Proposition. *Let C and D be halfgroups. Then*

(1) every homomorphism $\varphi \colon C \to D$ is an accumulator, and
(2) if $a, b \colon C \to C$ are accumulators, then so is their composition $a \circ b$.

Proof. Omitted. □

Points 1 and 2 of the preceding proposition imply that the subset of accumulators in the set of unary operators on a halfgroup form a monoid under composition of maps, with the identity map service as the identity for the monoid of accumulators. For a halfgroup C, let $\mathrm{Acc}(C)$ be the **monoid of accumulators** on C.

Definition. Let C and D be halfgroups. Let $\mathrm{Acc}(C, D)$ be the **set of accumulators** with domain C and codomain D. Let $a \colon C \to D$ be an accumulator. The **shift** of a by an element $c \in C$ is the unique map $b = a \leftarrow c \colon C \to D$ such that $a(cd) = a(c)b(d)$ for all $d \in C$.

Proposition. *Let C and D be halfgroups. If $a\colon C \to D$ is an accumulator, then*

(1) $a \leftarrow c$ is an accumulator for all $c \in C$,
(2) C acts on $\mathrm{Acc}(C, D)$ on the right by shift, and
(3) $a = a \leftarrow c$ for all $c \in C$ iff a is a homomorphism.

Proof. Points 1 and 3 are left to the reader as an exercise. For point 2, first note that $a \leftarrow 1 = a$ for all accumulators. Let $c, d \in C$. Let $b = a \leftarrow c$, $e = a \leftarrow (cd)$, and $f = b \leftarrow d$ For all $g \in C$, we have $a(cg) = a(c)b(g)$, $a(cdg) = a(cd)e(g)$, and $b(dg) = b(d)f(g)$. This gives $a(cdg) = a(cd)e(g) = a(c)b(d)e(g)$ and $a(cdg) = a(c)b(dg) = a(c)b(d)f(g)$, so by cancellation we obtain $e(g) = f(g)$, hence $a \leftarrow (cd) = (a \leftarrow c) \leftarrow d$. □

Definition. Let C be a halfgroup and $a\colon C \to C$ an accumulator. The **conjugate** of a by the monoid automorphism $\varphi\colon C \to C$ is $\varphi \circ a \circ \varphi^{-1}$. Let $\mathrm{Aut}(C)$ be the **group of monoid automorphisms** on C with multiplication composition of maps.

Proposition. *Let C be a halfgroup. Then $\mathrm{Aut}(C)$ acts on $\mathrm{Acc}(C)$ on the left by $\varphi \cdot a = \varphi \circ a$ and $\varphi \cdot a = a \circ \varphi^{-1}$ and on the right by $a \cdot \varphi = a \circ \varphi$ and $a \cdot \varphi = \varphi^{-1} \circ a$.*

Proof. Omitted □

Lemma. *(Square lemma) Let C, D, E, and F be halfgroups, let $a\colon C \to D$ and $b\colon E \to F$ be accumulators, and let $\varphi\colon C \to E$ and $\psi\colon D \to F$ be homomorphisms such that $\psi \circ a = b \circ \varphi$. Then for all $c \in C$, we have $\psi \circ (a \leftarrow c) = (b \leftarrow \varphi(c)) \circ \varphi$.*

Proof. Let $c, d \in C$, and let $e = a \leftarrow c$ and $f = b \leftarrow \varphi(a)$. Then we have $\psi(a(cd)) = \psi(a(c)e(d)) = \psi(a(c))\psi(e(d)) = b(\varphi(c))\psi(e(d))$ and $\psi(a(cd)) = b(\varphi(cd)) = b(\varphi(c)\varphi(d)) = b(\varphi(c))f(\varphi(d))$, so cancelling by $b(\varphi(c))$ on the left yields $\psi(e(d)) = f(\varphi(d))$.

[Note: the shifted accumulators $a \leftarrow c$ and $b \leftarrow \varphi(c)$ also satisfy the hypothesis of the square lemma.] □

Exercise. Let C be a halfgroup, $a\colon C \to C$ an accumulator, and $\varphi\colon C \to C$ a homomorphism. Prove that $(a \circ \varphi) \leftarrow c = (a \leftarrow \varphi(c)) \circ \varphi$ for all $c \in C$.

Networks on directed graphs

Definition. Let X be a directed graph. A **network** on X is a triple (C, r, a_0) such that $C\colon X' \to \text{Halfgroups}$, i.e. $C(x')$ is a halfgroup for all $x' \in X'$, $r = (r_{x'})_{x' \in X'}$, is the **route**, an indexed collection of maps $r(x', \cdot)\colon C_{x'} \to X$ such that $\perp(r(x', c)) = x'$ for all $x' \in X'$ and $c \in C(x')$, and $a_0\colon X \to \text{Accumulators}$, $a_0(x, \cdot)\colon C(\perp(x)) \to C(\uparrow(x))$ is the **link** along edge x.

Given a directed graph X and network (C, r, a_0) on X, we can calculate a single step of the **load** (x'_0, c_0)—where $c_0 \in C(x'_0)$, is the **content** and $x'_0 \in X'$ is the **location**—and the successor link accumulators as it passes through the network according to the route r as follows:

(1) the route map selects the edge $x_0 = r(x'_0, c_0)$ the load will travel along and gives the successor location $x'_1 = \uparrow(r(x'_0, c_0))$,
(2) the link $a_0(x_0, \cdot)\colon C(x'_0) \to C(x'_1)$ gives the successor content $c_1 = a_0(x_0, c_0)$, and

(3) as the load traverses the edge x_0, it erodes the link $a_0(x_0, \cdot)$, and we calculate the successor links a_1 as follows:

$$\begin{cases} a_1(y, \cdot) = a_0(y, \cdot) & \text{if } y \neq x_0 \\ a_1(x_0, \cdot) = a_0(x_0, \cdot) \leftarrow c_0 \end{cases}.$$

Iterating, this load yields the **traversal sequence** (x'_n, c_n, x_n) and **erosion sequence** a_n for all $n \in \mathbb{N}$.

The preceding description is for a network with a single load. To expand this to multiple loads, let $a = a_0$, require every link $a(x, \cdot)$ to be a homomorphism, and let I be an index set. Since shift preserves homomorphisms, the links $a(x, \cdot)$ do not change as loads traverse the network. To each $i \in I$ we assign a load (x'_{i0}, c_{i0}). To calculate the traversal sequence, we let $x'_{i1} = \uparrow (r(x'_{i0}, c_{i0}))$, $x_{i0} = r(x'_{i0}, c_{i0})$, $c_{i1} = a(x_{i0}, c_{i0})$, and iterate, forming the **multi-load traversal sequence** $(x'_{in}, c_{in}, x_{in})_{i \in I}$ for all $n \in \mathbb{N}$.

Definition. Let X and Y be directed graphs, $\gamma \colon X \to Y$ an injective map from the edges of X to the edges of Y, and $\gamma' \colon X' \to Y'$ a map from the nodes X' to the nodes Y'. Suppose that γ is compatible with the base and head maps, i.e. for all edges $x \in X$, we have $\gamma'(\perp (x)) = \perp (\gamma(x))$ and $\gamma'(\uparrow (x)) = \uparrow (\gamma(x))$. Let (C, r, a) be a network on X and (D, s, b) a network on Y. A **homomorphism** of networks from X to Y adapted to the edge and node maps (γ, γ') is an indexed collection $(\varphi_{x'})_{x' \in X'}$ of homomorphisms $\varphi(x', \cdot) \colon C(x') \to C(\gamma'(x'))$ that are compatible with the route maps and link accumulators, i.e. for all $x' \in X'$ and $c \in C(x')$, we have

(1) $\gamma(r(x', c)) = s(\gamma'(x'), \varphi(x', c))$, and
(2) for all $x \in X$ and $c \in C(\perp (x))$, we have $\varphi(\uparrow (x), a(x, c)) = b(\gamma(x), \varphi(\perp (x), c))$.

Theorem. *Let $X, Y, \gamma, \gamma', (C, r, a), (D, s, b)$, and $(\varphi_{x'})_{x' \in X'}$ be as above, and let (x'_0, c_0) be a load for (C, a, r). The operation of applying the network homomorphism φ commutes with calculating the routes and traversal sequence for the load.*

Proof. We must prove that if $y'_0 = \gamma'(x'_0)$, $d_0 = \varphi(x'_0, c_0)$, (x'_n, c_n, x_n) is the (C, r, a) traversal sequence for the load (x'_0, c_0), and (y'_n, d_n, y_n) is the (D, s, b) traversal sequence for the load (y'_0, d_0), then for all n, we have $y'_n = \gamma'(x'_n)$, $y_n = \gamma(x_n)$, and $d_n = \varphi(x'_n, c_n)$.

Let $\psi = \varphi(x'_0, \cdot)$, $\theta = \varphi(x'_1, \cdot)$, $e = a(x_0, \cdot)$, $f = b(y_0, \cdot)$, $E = C(x'_0)$, $F = C(x'_1)$, $G = D(y'_0)$, and $H = D(y'_1)$. First, note that E, F, G, and H together with ψ, θ, e, and f satisfy the hypothesis for the square lemma. Consider the traversal of the first link: before traversal, each pair of links x and $y = \gamma(x)$ together with the node homomorphisms $\varphi(\perp (x), \cdot)$ and $\varphi(\uparrow (x), \cdot)$ along with $C(\perp (x)), C(\uparrow (x)), D(\perp (y))$, and $D(\uparrow (y))$ satisfy the hypothesis for the square lemma. Since γ is injective, traversal erosion, i.e. replacing $a(x_0, \cdot)$ with $a(x_0, \cdot) \leftarrow c_0$ and $b(y_0, \cdot)$ with $b(y_0, \cdot) \leftarrow \varphi(x'_0, c)$ preserves this condition: for $x = x_0$ and $y = y_0$, this follows from the square lemma, and for $x \neq x_0$, this follows trivially. The commutativity conditions (points 1 and 2 in the definition above) establish the first step. The proof by induction is left to the reader as an exercise. □

The edge map γ in the last theorem must be injective, or else the conclusion does not hold:

Example. Let X be the directed graph with a single node x' and edges x_n indexed by $\{0, 1, 2, ...\}$, and let (C, r, a) be the following network on it: define $C(x') = \mathbb{N}$, let the route map be $r(x', i) = x_i$ with links $a(x_n, i) = i^2$. Since a route is never traversed more than once, the content of the traversal sequence started with load $(x', 2)$ is 2^{2^n}. Now let Y be the directed graph with single node y' and single edge y, and let (D, s, b_0) be the following network on it: define $D(y') = \mathbb{N}$, route map $s(y', i) = y$ with link $b_0(y, i) = i^2$. Define the node map $\gamma' \colon X' \to Y'$ to be $\gamma(x') = y'$, the edge map $\gamma \colon X \to Y$ to be $\gamma(x_n) = y$, and let $\varphi(x', \cdot) \colon C(x') \to D(y')$ be the identity map $\varphi(x', i) = i$. All of the conditions of the preceding theorem have been satisfied except injectivity of γ. For convenience, we calculate the shift of $f(x) = x^2$ here: for $t \in \mathbb{N}$, we have $(f \leftarrow t)(x) = (x+t)^2 - t^2 = x^2 + 2xt$. If $d_0 = 2$, the traversal of the load (y', d_0) can be calculated as follows: $d_1 = 4$, $b_1(y, \cdot) = b_0(y, \cdot) \leftarrow 2$, $d_2 = (b_0(y, \cdot) \leftarrow 2)(4) = 16+16 = 32$, $b_2(y, \cdot) = (b_0 \leftarrow 6)$, $d_3 = (b_0(y, \cdot) \leftarrow 6)(32) = 1408$, so traversal does not commute with the homomorphism $\varphi(x', \cdot)$. Intuitively, this is because traversal in X erodes different links, but traversal in Y cumulatively erodes the same link.