# GEOMETRY 1: real numbers.

You are supposed to know what field is, consult ALGEBRA-1 for the definition.

## Cauchy sequences.

Real numbers are usually considered as something that can be approximated by rational numbers, for example, one can regard a real number $a$ as infinite decimal fraction $a_0, a_1a_2\ldots$, the finite fragments of that fraction $a_0, a_1a_2\ldots a_n$ are then approximations of $a$. Some fractions are declared equivalent, for example, $1,00000\ldots$ and $0,9999\ldots$. It turns out that it is easier to rigorously define real numbers and operations on them when not only decimal fractions but just any sequences of rational numbers which approximate a given real number are considered. And again it should be taken into account that different sequences can be equivalent (when they approximate one and the same number). It appears quite logical to **define** a real number as a set of sequences of rational numbers that approximate it. This is Cauchy approach to real numbers definition.

**Definition 1.1.** We will say that something holds for **almost all** elements of a set if it holds for all elements except finite number of them. Let $\{a_i\} = a_0, a_1, a_2, \ldots$ be a sequence of rational numbers. It is said that $\{a_i\}$ is a **Cauchy sequence** if for any rational number $\varepsilon > 0$ there exists an interval $[x, y]$ of length $\varepsilon$ which contains almost all $\{a_i\}$.

**Exercise 1.1.** Let $a$ be a rational number. Prove that a constant sequence $a, a, \ldots$ is a Cauchy sequence.

We will denote such a sequence by $\{a\}$.

**Exercise 1.2.** Let $\{a_i\}$ be a Cauchy sequence. Let us permute arbitrarily its elements $a_i$. Prove that we obtain a Cauchy sequence then.

**Exercise 1.3.** Consider a sequence $\{a_i\}$ of rational numbers from an interval $I = [a, b]$, $a, b \in \mathbb{Q}$. Prove that one can select a subsequence out of $\{a_i\}$ which is a Cauchy subsequence.

**Hint.** Let us split the interval $I_0 = [a, b]$ into two equal parts. One of the halves (we will denote it by $I_1$) contains an infinite number of elements of the sequence. Let us delete from $\{a_i\}$ all elements that do not belong to $I_1$ except $a_0$. Let then divide $I_1$ into two equal parts and repeat the procedure over and over again. An interval $I_k$ obtained on a $k$-th step contains all elements of the sequence starting from $k$-th and this interval is of length $\frac{b-a}{2^k}$.

**Exercise 1.4 (!).** Consider a monotonically increasing sequence $a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots$. All $a_i$ are bounded by some constant $C$: $a_i \leqslant C$. Prove that this is a Cauchy sequence.

**Hint.** Use the previous problem.

**Definition 1.2.** Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences. They are called **equivalent** if a sequence $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$ is a Cauchy sequence.

**Exercise 1.5.** Let $a$, $b$ be two rational numbers. Prove that $\{a\}$ is equivalent to $\{b\}$ iff $a = b$.

**Exercise 1.6.** Prove that a Cauchy sequence is equivalent to any subsequence of it.

**Exercise 1.7.** Prove that if $\{a_i\}$ is equivalent to $\{b_i\}$ then $\{b_i\}$ is equivalent to $\{a_i\}$.

**Exercise 1.8 (!).** Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences. Prove that there exist two non-intersecting intervals $I_1$, $I_2$ such that almost all $a_i$ belong to $I_1$ while almost all $b_i$ belong to $I_2$.

**Hint.** Apply the definition of a Cauchy sequence with $\varepsilon = \frac{1}{2^n}$ for all $n$.

**Exercise 1.9 (!).** Prove that if a sequence $\{a_i\}$ is equivalent to a sequence $\{b_i\}$ and a sequence $\{b_i\}$ is equivalent to a sequence $\{c_i\}$ then $\{a_i\}$ is equivalent to $\{c_i\}$ (one says that "Cauchy sequences equivalence is transitive").

**Definition 1.3.** Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences. It is said that $\{a_i\} > \{b_i\}$ if $a_i > b_i$ for almost all $i$.

**Exercise 1.10.** Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences. Prove that either $\{a_i\} < \{b_i\}$ or $\{b_i\} < \{a_i\}$.

**Hint.** Use the problem 1.8.

**Exercise 1.11.** Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences and $\{a_i\} < \{b_i\}$. Prove that there exist two rational numbers $c$, $d$ such that $\{a_i\} < \{c\} < \{d\} < \{b_i\}$.

**Hint.** Use the previous hint.

**Exercise 1.12.** Let $\{a_i\} < \{b_i\}$ and $\{b_i\}$ be equivalent to $\{c_i\}$. Prove that $\{a_i\} < \{c_i\}$.

**Hint.** Use the previous problem and the definition of Cauchy sequence for $\varepsilon < |c - d|$.

**Exercise 1.13.** Let $\{a_i\}$ be a Cauchy sequence and $c \in \mathbb{Q}$ be a rational number. Prove that the following properties are equivalent

a. $\{a_i\}$ is equivalent to a sequence $\{c\}$.

b. there are infinitely many elements of a sequence $\{a_i\}$ in any open interval $]x, y[$ containing $c$.

c. any open interval $]x, y[$ which contains $c$ contains almost all elements of a sequence $\{a_i\}$ as well.

**Definition 1.4.** If any of these properties holds then it is said that $\{a_i\}$ converges to $c$.

**Exercise 1.14.** Let $\{a_i\}$, $\{b_i\}$ be a Cauchy sequence. Prove that $\{a_i + b_i\}$ and $\{a_i - b_i\}$ are Cauchy sequences.

**Exercise 1.15.** Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences and $b_i$ converges to 0. Prove that $\{a_i\}$ is equivalent to $\{a_i + b_i\}$.

**Exercise 1.16.** Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences. Prove that $\{a_i b_i\}$ is a Cauchy sequence.

**Exercise 1.17.** Prove that if $\{b_i\}$ converges to 1 then $\{a_i b_i\}$ is equivalent to $\{a_i\}$.

**Exercise 1.18.** Let $\{a_i\}$ be a Cauchy sequence which does not contain zeros and which does not converge to 0. Prove that $\{a_i^{-1}\}$ is a Cauchy sequence.

**Hint.** Prove that there exists a closed interval $[x, y]$ which does not contain 0 such that almost all $\{a_i\}$ are contained in $[x, y]$. Let almost all $\{a_i\}$ belong to an interval $I \subset [x, y]$ of a length $\varepsilon$. Prove that all $\{a_i^{-1}\}$ except a finite number belong to an interval $I^{-1}$ of a length $\varepsilon(\min(|x|, |y|)^{-1}$.

**Definition 1.5.** A set of all Cauchy sequences equivalent to a Cauchy sequence $\{a_i\}$ is called an **equivalence class** of a Cauchy sequence. The set of all equivalence classes is called a **set of real numbers** and is denoted by $\mathbb{R}$.

**Exercise 1.19.** Prove that to correspondence $c \mapsto \{c\}$ defines an injective mapping from a set $\mathbb{Q}$ of all rational numbers into $\mathbb{R}$.

**Exercise 1.20 (!).** Prove that four arithmetic operations that we have defined on $\mathbb{R}$ in the problems 1.14- 1.18 define on $\mathbb{R}$ the structure of a field.

## Dedekind sections

The main disadvantage of defining real numbers using Cauchy sequences is that there are too many Cauchy sequences and the definition appears to be too implicit. This difficulty is rather psychological. Nevertheless, there exists a way to overcome it, it is to introduce more straightforward definition of real numbers that was proposed by Dedekind.

**Definition 1.6.** Let $R \subset \mathbb{Q}$ be a subset of a set of rational numbers which is non-empty and does not equal to the whole $\mathbb{Q}$. It is said that $R$ is a **Dedekind section** if $a \in R$ and $b < a$ entails that $b \in R$. Dedekind section $R$ is said to be **closed** if there exists a rational number $a$ such that $b \in R$ as soon as $b \leqslant a$. Otherwise $R$ is said to be **open**.

Let $\{a_i\}$ be a Cauchy sequence. Let us denote the set of all rational numbers $b$ such that $\{b\} < \{a_i\}$ by $R_{\{a_i\}}$.

**Exercise 1.21.** Prove that $R_{\{a_i\}}$ is a Dedekind section (i.e. if $b \in R_{\{a_i\}}$ and $c < b$ then $c \in R_{\{a_i\}}$). Prove that this section is open.

**Exercise 1.22.** Let $\{a_i\}$ and $\{b_i\}$ be equivalent Cauchy sequences. Prove that $R_{\{a_i\}} = R_{\{b_i\}}$.

**Exercise 1.23.** Let $\{a_i\}$ and $\{b_i\}$ be non-equivalent Cauchy sequences and $\{a_i\} < \{b_i\}$. Prove that $R_{\{a_i\}} \subset R_{\{b_i\}}$ but those two sets do not coincide.

**Hint.** Consider the points of an interval $[c, d]$ from the problem 1.11; which of the sets $R_{\{a_i\}}$, $R_{\{b_i\}}$ do they belong?

**Exercise 1.24 (*).** Let $\{a_i\}$, $\{b_i\}$ be two Cauchy sequences. Prove that they are equivalent if and only if $R_{\{a_i\}} = R_{\{b_i\}}$.

**Hint.** Use the problem 1.10 (as well as the preceding problems).

**Exercise 1.25 (*).** Let $R \subset \mathbb{Q}$ be an open Dedekind section. Prove that $R = R_{\{a_i\}}$ holds for some Cauchy sequence $\{a_i\}$.

**Hint.** Consider an interval $I_0 = [a, b]$ such that $a$ belongs to $R$ and $b$ does not. Split it into two equal parts, select the half $I_1$ with the same property. Repeat this process and select any point of $I_i$ as $a_i$.

We observe that the set of equivalence classes of Cauchy sequences is the same thing as the set of open Dedekind sections. That is why the real numbers can be defined as Dedekind sections. In what follows you can use the definition that suits you best.

**Exercise 1.26 (**).** Define arithmetic operations on $\mathbb{R}$ explicitly on Dedekind sections without using Cauchy sequences. Check that the axioms of a field hold.

**Hint.** To define multiplication define first the operations "multiplication by a positive real number $a$" and "multiplication by $-1$", then prove distributivity for each of them separately.

## Supremum and infimum

**Definition 1.7.** Let $A \subset \mathbb{R}$ be some subset of $\mathbb{R}$. A set $A$ is said to be **bounded above** if all elements of $A$ are greater that some constant $C \in \mathbb{R}$. A set $A$ is said to be **bounded below** if all elements of $A$ are less than some constant $C \in \mathbb{R}$. A set $A$ is called **bounded** if it bounded above and bounded below.

**Definition 1.8.** Let $A \subset \mathbb{R}$ be some subset of $\mathbb{R}$. Infimum of $A$ (notation: $\inf A$) is by definition a number $c \in \mathbb{R}$ such that $c \leqslant a$ for all $a \in A$ and in any open interval $]x, y[$ containing $c$ there are elements of $a$. Supremum of $A$ (notation: $\sup A$) is by definition a number $c \in \mathbb{R}$ such that $c \geqslant a$ for all $a \in A$ and in any open interval $]x, y[$ containing $c$ there are elements of $a$.

**Exercise 1.27.** Prove that $\inf A$ and $\sup A$ are unique (if they exist).

**Exercise 1.28 (!).** Let $A$ be a set bounded above. Prove that $\sup A$ exists.

**Hint.** Consider every $a \in A$ as Dedekind sections, i.e. subsets of $\mathbb{Q}$. Consider their union $R$; since every $a \leqslant C$ this will be Dedekind section too. Prove that $\inf A = R$.

**Exercise 1.29 (!).** Let $A \subset \mathbb{R}$ a set bounded below. Prove that $\inf A$ exists.

**Remark.** Let $A \subset \mathbb{R}$ is not bounded above (below). It is denoted by $\inf A = -\infty$ ($\sup A = \infty$).

# GEOMETRY 2: real numbers, part 2.

## Roots of polynomials of an odd degree.

**Exercise 2.1 (!).** Consider a polynomial over $\mathbb{Q}$ of an odd degree, $P = t^{2n+1} + a_{2n}t^{2n} + a_{2n-1}t^{2n-1} + \dots + a_0$. Let $R_P$ be the set of all $x \in \mathbb{Q}$ such that $P(t) < 0$ on an interval $]-\infty, x]$. Prove that $R_P$ is not empty.

**Hint.** Prove that $R_P$ contains $-\max(1, \sum |a_i|)$.

**Exercise 2.2 (!).** Prove that $R_P$ is not the set of all real numbers.

**Hint.** Prove that the complement $\mathbb{Q} \backslash R_P$ contains $\max(1, \sum |a_i|)$.

**Exercise 2.3 (!).** Prove that $R_P$ is a Dedekind section.

**Exercise 2.4 (!).** Prove that $P$ satisfies **Lipschitz property**: for any interval $I$ there exists a constant $C > 0$ such that $|P(a) - P(b)| < C|a - b|$ for any $a, b \in I$.

**Exercise 2.5 (!).** Consider a Dedekind section $R_P$ as a real number. Prove that $P(R_P) = 0$. It follows that any polynomial over $\mathbb{R}$ of an odd degree has a root.

**Hint.** First prove that $P(R_P) \leqslant 0$. Then prove that $P(R_P) < 0$ contradicts the problem 2.4.

## Limits.

**Definition 2.1.** Let $A \subset \mathbb{R}$ be a set of real numbers and $c$ be a real number. Then $c$ is called **accumulation point (limit point)** of a set $A$ if every open interval $I = ]x, y[$ containing $c$ contains infinitely many elements of $A$.

**Definition 2.2.** Let $\{a_i\}$ be a sequence of real numbers and $c$ be a real number. Let any open interval $I = ]x, y[$ containing $c$ contain all elements of $\{a_i\}$ except a finite number of them. Then $c$ is called the **limit of the sequence** $\{a_i\}$ (denoted by $c = \lim\limits_{i \to \infty} a_i$). It is said that a sequence $a_i$ **converges to** $c$.

**Exercise 2.6.** Let $c$ be an accumulation point of a sequence $\{a_i\}$. Prove that there exists a subsequence of $\{a_i\}$ that converges to $c$.

**Exercise 2.7 (*).** Consider a sequence $\{a_i\}$ of points from an interval $[x, y]$. Prove the existence of accumulation points of that sequence.

**Definition 2.3.** A set $A \subset \mathbb{R}$ is called **discrete** if it has no accumulation points.

**Exercise 2.8 (*).** Let $\{a_i\}$ be a sequence. Denote a set of all $a_i$ by $A$. Prove that $\{a_i\}$ converges if and only if $A$ has no infinite discrete subsets and has a unique accumulation point.

**Exercise 2.9.** Consider a sequence $0, 1, 2, 3, 4, \dots$. Prove that this sequence has no limit.

**Exercise 2.10.** Consider a sequence $0, 1, 1/2, 1/3, 1/4, \dots$. Prove that this sequence converges to 0.

**Exercise 2.11.** Consider an increasing sequence $a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots$, $a_i \in \mathbb{R}$. Let all $a_i$ be bounded above by $C$: $a_i \leqslant C$. Prove that this sequence has a limit. Use the definition of real numbers as Dedekind sections.

**Hint.** Prove that $\lim\limits_{i\to\infty} a_i = \sup\{a_i\}$, and use the fact that the supremum exists.

**Definition 2.4.** Let $\{a_i\} = a_0, a_1, a_2, \ldots$ be a sequence of real numbers. $\{a_i\}$ is called a **Cauchy sequence** if for any $\varepsilon > 0$ there exists an interval $[x, y]$ of length $\varepsilon$ which contains all members $\{a_i\}$ except a finite number of them.

**Remark.** This is the same definition as the definition of Cauchy sequences of rational numbers.

**Exercise 2.12.** Let a sequence $\{a_i\}$ converge to a real number $c$. Prove that this is a Cauchy sequence.

**Exercise 2.13.** Let a Cauchy sequence $\{a_i\}$ have a subsequence that converges to $x \in \mathbb{R}$. Prove that $\{a_i\}$ converges to $x$.

**Exercise 2.14.** Let $\{a_i\}$ be a Cauchy sequence. Consider the sequence $\{b_i\}$, $b_i = \inf_{i\geqslant k} a_i$. Prove that this infimum is correctly defined and that the sequence $b_i$ increases.

**Exercise 2.15.** Consider the previous problem and prove that if the sequence $\{b_i\}$ has a limit then $\lim\limits_{i\to\infty} a_i = \lim\limits_{i\to\infty} b_i$.

**Exercise 2.16 (!).** Let $\{a_i\}$ be a Cauchy sequence. Prove that $\{a_i\}$ converges. Use the definition of real numbers as Dedekind sections.

**Hint.** Use the previous problem.

**Exercise 2.17 (!).** Let $\{a_i\}$ be a Cauchy sequence. Prove that $\{a_i\}$ converges. Use the definition of real numbers as Cauchy sequences.

**Hint.** Let a real number $\{a_i\}$ be represented by a Cauchy sequence of rational numbers $a_i(0)$, $a_i(1)$, $a_i(2), \ldots$. Passing to a suitable subsequence one can suppose that all $a_i$ $(i > n)$ are contained in an interval on length $2^{-n}$ and that all $a_i(j)$ $(j > m)$ are contained in an interval of length $2^{-m}$. Prove that the sequence $\{a_i(i)\}$ is a Cauchy sequence and that the sequence $\{a_i\}$ converges to the real number represented by it.

**Exercise 2.18 (!).** Let $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ be converging sequences of real numbers and $a_i \leqslant b_i \leqslant c_i$ for all $i$. Suppose that $\lim\limits_{i\to\infty} a_i = \lim\limits_{i\to\infty} c_i = x$. Prove that $\lim\limits_{i\to\infty} b_i = x$.

**Exercise 2.19 (*).** Let a sequence $\{a_i\}$ converge to $x$. Prove that $b_j = \frac{1}{j}\sum_{i=0}^{j} a_i$ converges to $x$. Give an example, when $\{b_j\}$ converges but $\{a_i\}$ does not.

## Series.

**Definition 2.5.** Let $\{a_i\}$ be a sequence of real numbers. Consider a sequence of partial sums $\sum_{i=0}^{n} a_i$. If this sequence converges it is said that **series** $\sum_{i=0}^{\infty} a_i$ **converge**. It is denoted by $\sum_{i=0}^{\infty} a_i = x$ where

$$x = \lim_{i\to\infty} \sum_{i=0}^{n} a_i.$$

It is often denoted by $\sum a_i = x$.

**Definition 2.6.** A series $\sum a_i$ **converges absolutely**, if series $\sum |a_i|$ converges.

**Exercise 2.20 (!).** Consider a series $\sum a_i$ which converges absolutely. Prove that these series converges.

**Exercise 2.21.** Consider a series $\sum a_i$ which converges absolutely. Let $b_i$ be a sequence of non-negative numbers such that $a_i \geqslant b_i$. Prove that the series $\sum b_i$ converges absolutely.

**Exercise 2.22 (**).** Let $a_i$, $b_i$ be sequences of integer numbers such that the series $\sum a_i^2$, $\sum b_i^2$ converge. Prove that the series $\sum a_i b_i$ converge.

**Exercise 2.23 (*).** Let $a_i$ be a sequence of positive real numbers. Limit of the sequence of products

$$\lim_{n\to\infty} \prod_{i=0}^{n} (1 + a_i)$$

is denoted by $\prod_{i=0}^{\infty}(1 + a_i)$. If this limit exists it is said that an infinite product $\prod_{i=0}^{\infty}(1 + a_i)$ converges. Let the product $\prod_{i=0}^{\infty}(1 + a_i)$ converge. Prove that the series $\sum_{i=0}^{\infty} a_i$ converges.

**Exercise 2.24 (*).** Prove that the infinite product $\prod_{i=0}^{\infty}(1 + \frac{1}{3^n})$ converges.

**Exercise 2.25 (**).** Let a series $\sum a_i$ converge. Prove that $\prod_{i=0}^{\infty}(1 + a_i)$ converges, as well.

**Exercise 2.26 (!).** Let $a_0 \geqslant a_1 \geqslant a_2 \geqslant \ldots$ be a decreasing sequence of positive real numbers converging to 0. Consider the series $\sum_{i=0}^{\infty}(-1)^i a_i$. Prove that these series converges. Such series are called **sign-alternating**.

**Exercise 2.27.** Prove that the series $\sum \frac{1}{n(n+1)}$ converges.

**Hint.** Consider the value $\frac{1}{n} - \frac{1}{(n+1)}$.

**Exercise 2.28.** Prove that the series $\sum \frac{1}{n^2}$ converges.

**Exercise 2.29.** Prove that the series $\sum \frac{1}{n!}$ converges.

**Exercise 2.30 (!).** Prove that the series $\sum \frac{1}{2^n}$ converges. Calculate the value it converges to.

**Exercise 2.31 (*).** Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.

**Exercise 2.32 (**).** Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ in a complete ordered field $A$. Does this series converge for all $x \in A$?

# GEOMETRY 3: Metric spaces and norm

You are supposed to know the definition of a linear space and dot product (i.e. a positive bilinear symmetric form). Consult ALGEBRA 3.

## Metric spaces, convex sets, norm

**Definition 3.1.** A metric space is a set $X$ equipped with a function $d : X \times X \to \mathbb{R}$ such that

a. $d(x, y) > 0$ for all $x \neq y \in X$; moreover, $d(x, x) = 0$.

b. Symmetry: $d(x, y) = d(y, x)$

c. "Triangle inequality": for all $x, y, z \in X$,

$$d(x, z) \leqslant d(x, y) + d(y, z).$$

A function $d$ which satisfies these conditions is called **metric**. The number $d(x, y)$ is called "distance between $x$ and $y$".

If $x \in X$ is a point and $\varepsilon$ is a real number then the set

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon$$

is called an **(open) ball of radius** $\varepsilon$ **with the center in** $x$. Such ball can be called as well an $\varepsilon$**-ball**. A closed ball is defined as follows

$$\overline{B}_\varepsilon(x) = \{y \in X \mid d(x, y) \leqslant \varepsilon.$$

**Exercise 3.1.** Consider any subset of a Euclidean plane $\mathbb{R}^2$ and the function $d$ defined as $d(a, b) = |ab|$ where $|ab|$ is the length of a segment $[a, b]$ on the plane. Prove that this defines a metric space.

**Exercise 3.2.** Consider the function $d_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

$$(x, y), (x', y') \mapsto \max(|x - x'|, |y - y'|).$$

Prove that this is a metric. Describe a unit ball with the center in zero.

**Exercise 3.3.** Consider a function $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

$$(x, y), (x', y') \mapsto |x - x'| + |y - y'|.$$

Prove that this is a metric. Describe a unit ball with the center in zero.

**Exercise 3.4 (*).** A function $f : [0, \infty[ \to [0, \infty[$ is said to be **upper convex** if $f(\frac{ax+by}{a+b}) \geqslant \frac{af(x)+bf(y)}{a+b}$, for any positive $a$, $b \in \mathbb{R}$. Let $f$ be such a function and $(X, d)$ be a metric space. Suppose that $f(\lambda) = 0$ iff $\lambda = 0$. Prove that the function $d_f(x, y) = f(d(x, y))$ defines a metric on $X$.

**Exercise 3.5.** Let $V$ be a linear space with a positive bilinear symmetric form $g(x, y)$ (in what follows we will call that form a **dot product**). Define the "distance" $d_g : V \times V \to \mathbb{R}$ as $d_g(x, y) = \sqrt{g(x - y, x - y)}$. Prove that $d(x, y) \geqslant 0$ where equality holds iff $x = y$.

**Definition 3.2.** Let $x \in V$ be a vector from a vector space $V$. **Parallel transport along vector** $x$ is a mapping $P_x : V \to V$, $y \mapsto y + x$.

**Exercise 3.6.** Prove that a function $d_g$ is "invariant with respect to parallel transports", i.e. $d_g(a, b) = d_g(P_x(a), P_x(b))$.

**Exercise 3.7.** Prove that if $y \neq 0$, then $d_g$ satisfies the triangle inequality:

$$\sqrt{g(x-y, x-y)} \leqslant \sqrt{g(x,x)} + \sqrt{g(y,y)}$$

**Hint.** Consider a two-dimensional subspace $V_0 \subset V$, generated by $x$ and $y$. Prove that it is isomorphic (as a space with dot product) to the space $\mathbb{R}^2$ with dot product $g((x, y), (x', y')) = xx' + yy'$. Use the triangle inequality for $\mathbb{R}^2$.

**Exercise 3.8 (!).** Prove that $d_g$ satisfies the triangle inequality.

**Hint.** Use invariance of parallel transports and reduce to the previous problem.

**Definition 3.3.** Consider a vector space $V$ with a dot product $g$, and let $d_g$ be the metric constructed above. This metric is called a **euclidean** metric.

**Definition 3.4.** Consider a vector space $V$, a parallel transport $P_x : V \to V$ and a one-dimensional subspace $V_1 \subset V$. Then the image $P_x(V_1)$ is called a **line** in $V$.

**Exercise 3.9.** Consider two different points in $x, y \in V$. Prove that there exists a unique line $V_{x,y}$ through $x$ and $y$.

**Definition 3.5.** Consider a line $l$ through points $x$ and $y$, and a point $a$ on $l$. We say that $a$ lies **between** $x$ and $y$, if $d(x, a) + d(b, y) = d(x, y)$. A **line segment between** $x$ **and** $y$ (denoted $[x, y]$) is the set of all points belonging to the line $V_{x,y}$, that "lie between" $x$ and $y$.

**Exercise 3.10.** consider three different points on a line. Prove that one (and only one) of these points lies between two other points. Prove that the line segment $[x, y]$ is a set of all points $z$ of the form $ax + (1 - a)y$, where $a \in [0, 1] \subset \mathbb{R}$.

**Definition 3.6.** Consider a vector space $V$, and let $B \subset V$ be its subset. We say that $B$ is **convex** if $B$ contains all points of the line segment $[x, y]$ for any $x, y \in V$.

**Definition 3.7.** Let $V$ be a vector space over $\mathbb{R}$. A **norm** on $V$ is a function $\rho : V \to \mathbb{R}$, such that the following hold:

a. For any $v \in V$ one has $\rho(v) \geqslant 0$. Moreover, $\rho(v) > 0$ for all nonzero $v$.

b. $\rho(\lambda v) = |\lambda|\rho(v)$

c. For any $v_1, v_2 \in V$ one has $\rho(v_1 + v_2) \leqslant \rho(v_1) + \rho(v_2)$.

**Exercise 3.11.** Consider a vector space $V$ over $\mathbb{R}$, and let $\rho : V \to \mathbb{R}$ be a norm on $V$. Consider the function $d_\rho : V \times V \to \mathbb{R}$, $d_\rho(x, y) = \rho(x - y)$. Prove that this is a metric on $V$.

**Exercise 3.12 (*).** Let $d : V \times V \to \mathbb{R}$ be a metric on $V$, invariant w.r.t. the parallel transports. Suppose that $d$ satisfies

$$d(\lambda x, \lambda y) = |\lambda| d(x, y)$$

for all $\lambda \in \mathbb{R}$. Prove that $d$ can be obtained from the norm $\rho : V \to \mathbb{R}$ by using the formula $d(x, y) = \rho(x - y)$.

**Exercise 3.13.** Let $V$ be a linear space over $\mathbb{R}$ and $\rho: V \to \mathbb{R}$ be a norm on $V$. Consider the set $B_1(0)$ of all points with the norm $\leqslant 1$. Prove that this set is convex.

**Definition 3.8.** Consider a vector space $V$ over $\mathbb{R}$ and let $v$ be a nonzero vector. Then the set of all vectors of the form $\{\lambda v \mid \lambda > 0\}$ is called a **half-line** (or a **ray**) in $V$.

**Definition 3.9.** A **central symmetry** in $V$ is the mapping $x \mapsto -x$.

**Exercise 3.14 (*).** Consider a central symmetric convex set $B \subset V$ that does not contain any half-lines and has an intersection with any half-line $\{\lambda v \mid \lambda > 0\}$. Consider the function $\rho$

$$v \overset{\rho}{\mapsto} \inf\{\lambda \in \mathbb{R}^{>0} \quad | \quad \lambda v \notin B\}$$

Prove that this is a norm on $V$. Prove that all the norms can be obtained that way.

**Exercise 3.15.** Consider an abelian group $G$ and a function $\nu: G \to \mathbb{R}$ satisfying $\nu(g) \geqslant 0$ for all $g \in G$, and $\nu(g) > 0$ whenever $g \neq 0$. Suppose that $\nu(a+b) \leqslant \nu(a) + \nu(b)$, $\nu(0) = 0$ and that $\nu(g) = \nu(-g)$ for all $g \in G$. Prove that the function $d_\nu: G \times G \to \mathbb{R}$, $d_\nu(x,y) = \nu(x-y)$ is a metric on $G$.

**Exercise 3.16.** A metric $d$ on an abelian group $G$ is called an **invariant** metric if $d(x+g, y+g) = d(x,y)$ for all $x, y, g \in G$. Prove that any invariant metric $d$ is obtained from a function $\nu: G \to \mathbb{R}$ by setting $d(x,y) = \nu(x-y)$.

**Definition 3.10.** Fix a prime number $p \in \mathbb{Z}$. The function $\nu_p: \mathbb{Z} \to \mathbb{R}$, which given a number $n = p^k r$ ($r$ is not divisible by $p$) yields $p^{-k}$, and satisfies $\nu_p(0) = 0$, is called the **$p$-adic norm on $\mathbb{Z}$**.

**Exercise 3.17.** Prove that the function $d_p(m,n) = \nu_p(n-m)$ defines a metric on $\mathbb{Z}$. This metric is called **$p$-adic metric on $\mathbb{Z}$**.

**Hint.** Check that $\nu_p(a+b) \leqslant \nu(a) + \nu(b)$ holds and use the previous problem.

**Definition 3.11.** Let $R$ be a ring and $\nu: R \to \mathbb{R}$ be a function that is positive and yields strictly positive values for all nonzero $r$. Suppose that $\nu(r_1 r_2) = \nu(r_1)\nu(r_2)$ and $\nu(r_1 + r_2) \leqslant \nu(r_1) + \nu(r_2)$. Then $\nu$ is called a **norm** on $R$. A ring endowed with a norm is called a **normed ring**.

**Remark.** It follows from the problems above that a norm on a ring $R$ defines an invariant metric on $R$. In what follows any normed ring will be regarded as a metric space.

**Exercise 3.18.** Prove that $\nu_p$ is a norm on a ring $\mathbb{Z}$. Define a norm on $\mathbb{Q}$ that extends $\nu_p$.

## Complete metric spaces.

**Definition 3.12.** Let $(X,d)$ be a metric space and $\{a_i\}$ be a sequence of point from $X$. A sequence $\{a_i\}$ is called a **Cauchy sequence**, if for every $\varepsilon > 0$ there exists an $\varepsilon$-ball in $X$ which contains all but a finite number of $a_i$.

**Exercise 3.19.** Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences in $X$. Prove that $\{d(a_i, b_i)\}$ is a Cauchy sequence in $\mathbb{R}$.

**Definition 3.13.** Let $(X, d)$ be a metric space and $\{a_i\}$, $\{b_i\}$ be Cauchy sequences in $X$. Sequences $\{a_i\}$ and $\{b_i\}$ are called **equivalent**, if the sequence $a_0, b_0, a_1, b_1, \ldots$ is a Cauchy sequence.

**Exercise 3.20.** Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences in $X$. Prove that $\{a_i\}$, $\{b_i\}$ are equivalent iff $\lim_{i\to\infty} d(a_i, b_i) = 0$.

**Exercise 3.21.** Let $\{a_i\}$, $\{b_i\}$ be equivalent Cauchy sequences in $X$, and $\{c_i\}$ be another Cauchy sequence. Prove that

$$\lim_{i\to\infty} d(a_i, c_i) = \lim_{i\to\infty} d(b_i, c_i)$$

**Exercise 3.22 (!).** Let $(X, d)$ be a metric space and $\overline{X}$ be the set of equivalence classes of Cauchy sequences. Prove that the function

$$\{a_i\}, \{b_i\} \mapsto \lim_{i\to\infty} d(a_i, b_i)$$

defines a metric on $\overline{X}$.

**Definition 3.14.** In that case, $\overline{X}$ is called the **completion of** $X$.

**Exercise 3.23.** Consider a natural mapping $X \to \overline{X}$, $x \mapsto \{x, x, x, x, ...\}$. Prove that it is an injection which preserves the metric.

**Definition 3.15.** Let $A$ be a subset of $X$. An element $c \in X$ is called an **accumulation point (limit point)** of a set $A$ if any open ball containing $c$ contains an infinite number of elements of $A$.

**Exercise 3.24.** Prove that a Cauchy sequence cannot have more than one accumulation point.

**Definition 3.16.** Let $\{a_i\}$ be a Cauchy sequence. It is said that $\{a_i\}$ **converges to** $x \in X$, or that $\{a_i\}$ **has the limit** $x$ (denoted as $\lim_{i\to\infty} a_i = x$), if $x$ is an accumulation point of $\{a_i\}$

**Definition 3.17.** A metric space $(X, d)$ is called **complete** if any Cauchy sequence in $X$ has a limit.

**Exercise 3.25 (!).** Prove that the completion of a metric space is complete.

**Definition 3.18.** A subset $A \subset X$ of a metric space is called **dense** if any open ball in $X$ contains an element from $A$.

**Exercise 3.26.** Prove that $X$ is dense in its completion $\overline{X}$.

**Exercise 3.27 (*).** Let $X$ be a metric space and consider a metric preserving mapping $j : X \to Z$ from $X$ into a complete metric space $Z$. Prove that $j$ can be uniquely extended to $\overline{j} : \overline{X} \to Z$.

**Remark.** This problem can be used as a definition of $\overline{X}$. The definition 3.14 then becomes a theorem.

**Exercise 3.28 (!).** Let $R$ be a ring endowed with a norm $\nu$. Define addition and multiplication on the completion of $R$ with respect to the metric corresponding to $\nu$. Prove that $\overline{R}$ has a norm that extends the norm $\nu$ on $R$.

**Definition 3.19.** The normed ring $\overline{R}$ is called the **completion of $R$ with respect to the norm $\nu$.**

**Exercise 3.29 (*).** Let $R$ be a normed ring and $\overline{R}$ be its completion. Suppose that $R$ is a field. Prove that $\overline{R}$ is also a field.

**Exercise 3.30 (*).** Let $R$ be a ring without zero divisors (i.e. it satisfies the following property: if $r_1$, $r_2$ are nonzero elements, then $r_1 r_2$ is also non-zero). Consider a function $\nu: R \to \mathbb{R}$ which maps all non-zero elements of $R$ to unity and maps zero to zero. Prove that $\nu$ is a norm. What is $\overline{R}$?

**Exercise 3.31.** Prove that $\mathbb{R}$ can be obtained as the completion of $\mathbb{Q}$ with respect to the norm $q \mapsto |q|$. Can this statement be used as a definition of $\mathbb{R}$?

**Definition 3.20.** The completion of $\mathbb{Z}$ with respect to the norm $\nu_p$ is called the **ring of integer $p$-adic numbers**. This ring is denoted by $\mathbb{Z}_p$.

**Exercise 3.32.** Let $(X, d)$ be a metric space and $\{a_i\}$ be a sequence of points in $X$. Suppose that the series $\sum d(a_i, a_{i-1})$ converges. Prove that $\{a_i\}$ is a Cauchy sequence. Is the converse true?

**Exercise 3.33 (!).** Prove that for any sequence of integer numbers $a_k$ the series $\sum a_k p^k$ converges in $Z_p$.

**Hint.** Use the previous problem.

**Exercise 3.34.** Prove that $(1-p)(\sum_{k=0}^{\infty} p^k) = 1$ in $Z_p$.

**Exercise 3.35 (*).** Prove that any integer number which is not divisible by $p$ is invertible in $Z_p$.

**Definition 3.21.** The completion of $\mathbb{Q}$ with respect to the norm obtained by extension of $\nu_p$, is denoted by $\mathbb{Q}_p$ and is called the **field of $p$-adic numbers**.

**Exercise 3.36 (*).** Take $x \in \mathbb{Q}_p$. Prove that $x = \frac{x'}{p^k}$, where $x' \in \mathbb{Z}_p$.

**Exercise 3.37 (*).** Prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

**Definition 3.22.** A norm $\nu$ on a ring $R$ is called **non-Archimedean**, if $\nu(x+y) \leqslant \max(\nu(x), \nu(y))$ for all $x, y$. Otherwise the norm is called **Archimedean**.

**Exercise 3.38 (*).** Let $\nu$ be a norm on $\mathbb{Q}$. Prove that $\nu$ is non-Archimedean iff $\mathbb{Z}$ is contained in the unit ball.

**Hint.** Use the following equality: $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. Find an estimate of $\sqrt[n]{((\nu(x+y)^n)}$ for big $n$ using the estimate of binomial coefficients: $\nu(\binom{k}{n}) \leqslant 1$.

**Exercise 3.39 (*).** Let $\nu$ be a non-Archimedean norm on $\mathbb{Q}$. Consider $\mathfrak{m} \subset \mathbb{Z}$ consisting of all integers $n$ such that $\nu(n) < 1$. Prove that $\mathfrak{m}$ is an *ideal* in $\mathbb{Z}$ (ideal in a ring $R$ is a subset which is closed under addition and multiplication by elements of $R$). Prove that the ideal

$$\mathfrak{m} = \{n \in \mathbb{Z} \mid \nu(n) < 1\}$$

is *prime* (prime ideal is an ideal such that $xy \notin \mathfrak{m}$ for all $x, y \notin \mathfrak{m}$).

**Exercise 3.40 (*).** Prove that any ideal in $\mathbb{Z}$ is of the form $\{0, \pm 1m, \pm 2m, \pm 3m, ...\}$ for some $m \in \mathbb{Z}$. Prove that any prime ideal $\mathfrak{m}$ in $\mathbb{Z}$ is of the form $\{0, \pm p, \pm 2p, \pm 3p, ...\}$, where $p = 0$ or $p$ prime.

**Hint.** Use the Euclid's Algorithm.

**Exercise 3.41 (*).** Let $\nu$ be a non-Archimedean norm $\mathbb{Q}$ and $\mathfrak{m} = \{p, 2p, 3p, 4p, ...\}$ be an ideal constructed above. Prove that there exists a real number $\lambda > 1$ such that $\nu(n) = \lambda^{-k}$ for any $n = p^k r$, $r \not| p$.

**Exercise 3.42 (*).** Let $\nu$ be a norm on $\mathbb{Q}$ such that $\nu(2) \leqslant 1$. Prove that $\nu(a) < \log_2(a) + 1$ for any integer $a > 0$.

**Hint.** Use the binary representation of a number.

**Exercise 3.43 (*).** Let $\nu$ be a norm on $\mathbb{Q}$ such that $\nu(2) < 1$. Prove that $\nu(a) \leqslant 1$ for any integer $a > 0$ (i.e. $\nu$ is non-Archimedean).

**Hint.** Deduce from $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ that $\lim\limits_{n\to\infty} \frac{\log n}{n} = 0$. Prove $\lim\limits_{N\to\infty} \nu(a^N) \leqslant 1$, using the previous problem.

**Exercise 3.44 (*).** Let $a_i$ be a Cauchy sequence of rational numbers of the form $\frac{x}{2^n}$ ("Cauchy sequence" here means the same thing as Cauchy sequence of real numbers). Suppose that a norm $\nu$ on $\mathbb{Q}$ is Archimedean. Prove that $\nu(a_i)$ is a Cauchy sequence.

**Hint.** Write down $x$ in the binary system and prove that

$$\nu(x/2^n) \leqslant \nu(2)^{\log_2(x)+1}/\nu(2)^n \leqslant \nu(2)^{\log_2|x+1/2^n|}.$$

**Exercise 3.45 (*).** Deduce that $\nu$ can be extended to a continuous function on $\mathbb{R}$, which satisfies $\nu(xy) = \nu(x)\nu(y)$. Prove that $\nu$ can be obtained as $x \mapsto |x|^\lambda$ for some constant $\lambda > 0$. Express $\lambda$ in terms of $\nu(2)$.

**Exercise 3.46 (*).** For which $\lambda > 0$ the function $x \mapsto |x|^\lambda$ defines a norm on $\mathbb{Q}$?

We have obtained a complete classification of norms on $\mathbb{Q}$: any norm can be obtained as a power of either a $p$-adic norm or the absolute value norm. This classification is called **Ostrovsky theorem**.

# GEOMETRY 4: Topology of metric spaces.

**Definition 4.1.** Let $M$ be a metric space and $X \subseteq M$. Then $X$ is called **open** when it contains, together with any point $x \in X$, some $\varepsilon$-ball with the center in $x$. A subset is called **closed** if its complement is open.

**Exercise 4.1.** Prove that $X$ is open iff for any sequence $\{a_i\}$ converging to $x \in X$ all but a finite number of $a_i$ belong to $X$.

**Exercise 4.2.** Prove that the union of any number of open sets is open. Prove that the intersection of a finite number of closed sets is closed.

**Exercise 4.3.** Prove that the closed ball

$$\overline{B}_\varepsilon(x) = \{y \in X \ \mid \ d(x,y) \leqslant \varepsilon\}$$

is a closed subset.

**Exercise 4.4.** Prove that a set is closed iff it contains all its accumulation points.

**Definition 4.2.** The **closure** of a set $A \subset M$ is the union of $A$ and the set of all the accumulation points of $A$.

**Exercise 4.5.** Consider a metric space, a closed ball $\overline{B}_\varepsilon(x)$ and an open ball $B_\varepsilon(x)$. Is it always true that $\overline{B}_\varepsilon(x)$ is the closure of $B_\varepsilon(x)$? Prove that the closure of any subset is always closed.

**Exercise 4.6.** Let $A$ be a subset of $M$ which has no accumulation points (such a subset is called **discrete**). Prove that $M \backslash A$ is open.

**Definition 4.3.** Let $M$ be a metric space and $\varepsilon > 0$ be a number. Consider $R \subseteq M$ such that $M$ can be covered by a union of all $\varepsilon$-balls with center in $R$. Then $R$ is called an **$\varepsilon$-net**.

**Exercise 4.7.** Let any sequence in $M$ have an accumulation point. Prove that for any $\varepsilon > 0$ in $M$ there exists a finite $\varepsilon$-net.

**Hint.** Suppose that there is no such net, then for any finite set $R$ there exists a point $x$, whose distance to $R$ is more than $\varepsilon$. Add $x$ to $R$, and, using this operation as induction step, obtain an infinite discrete subset of $M$.

**Definition 4.4.** Let $X \subset M$ and $U_i \subset M$ be a collection of open sets. If $X \subset \cup U_i$ then it is said that $U_i$ is a **cover of** $X$. A collection of sets obtained from $\{U_i\}$ by throwing out some open sets in such a way that it remains a cover, is called a **subcover**.

**Exercise 4.8.** Let $M$ be a metric space, $S$ be an open cover of $M$. Let every subsequence of elements of $M$ have an accumulation point. Prove that there exists such an $\varepsilon > 0$, that any ball of radius $< \varepsilon$ is contained in one of the sets of the cover $S$.

**Hint.** Suppose that for any $\varepsilon$ there exists a point $x_\varepsilon$ such that a corresponding $\varepsilon$-ball is not contained entirely in any of the sets of the cover. Consider a sequence $\{\varepsilon_i\}$ which converges to zero and let $x$ be an accumulation point of $\{x_{\varepsilon_i}\}$. Prove that $x$ is not contained in any of the sets of $S$.

**Exercise 4.9 (!).** (Bolzano-Weierstrass lemma) Let $X \subset M$ be a subset of a metric space. Prove that the following conditions are equivalent

a. Every sequence of points from $X$ has an accumulation point in $X$.

b. Every open cover of $X$ has a finite subcover.

**Hint.** Use problem 4.6 to deduce (a) from (b). In order to deduce (b) from (a), take an arbitrary cover $S$, a number $\varepsilon$ from the problem 4.8 and a finite $\varepsilon$-net. Every ball of the $\varepsilon$-net is contained in some of the elements $U_i \in S$. Prove that $\{U_i\}$ is a finite subcover.

**Definition 4.5.** Let $M$, $M'$ be metric spaces, and $f: M \to M'$ be a function. Then $f$ is called **continuous**, if $f$ maps any sequence that converges to $x$ to a sequence that converges to $f(x)$, for all $x \in M$.

**Exercise 4.10 (!).** Let $X$ be any subset of $M$. Prove that a function $f: M \to \mathbb{R}, x \stackrel{f}{\mapsto} d(\{x\}, X)$ is continuous, where $d(\{x\}, X)$ (distance between $x$ and $X$) is defined as $d(\{x\}, X) := \inf_{x' \in X} d(x, x')$.

**Definition 4.6.** Let $M$ be a metric space, $X \subset M$. It is said that $X$ is a **compact set**, if any of the statements of the problem 4.9 holds. Note that these conditions do not depend on inclusion $X \hookrightarrow M$, but only on the metric on $X$.

**Exercise 4.11 (!).** Consider the completion of $\mathbb{Z}$ with respect to the norm $\nu_p$ defined above (it is called "a ring of integer $p$-adic numbers" and is denoted $\mathbb{Z}_p$). Prove that it is compact.

**Hint.** Prove that any $p$-adic number can be represented in the from $\sum a_i p^i$, where $a_i$ are integers between 0 and $p-1$.

**Exercise 4.12.** Prove that a compact subset of $M$ is always closed.

**Hint.** Prove that it contains all its accumulation points.

**Exercise 4.13.** Prove that a closed subspace of a compact set is always compact.

**Exercise 4.14.** Prove that a union of a compact sets is compact.

**Exercise 4.15 (!).** Let $f: X \to \mathbb{R}$ be a continuous function defined on a compact set. Prove that $f$ achieves maximum on $X$.

**Definition 4.7.** Let $X, Y$ be two subsets of a metric space. Denote the number $\inf_{x \in X, y \in Y}(d(x, y))$ by $d(X, Y)$.

**Exercise 4.16 (!).** Let $X$, $Y$ be two compact subsets of a metric space. Prove that there exist points $x, y$ in $X, Y$ such that $d(x, y) = d(X, Y)$.

**Definition 4.8.** A subset $Z \subset M$ is called bounded if it is contained in a ball $B_r(x)$ for some $r \in \mathbb{R}, x \in M$.

**Exercise 4.17.** Let $Z \subset M$ be compact. Prove that it is bounded.

**Definition 4.9.** Let $M$ be a metric space and $X \subset M$. The union of all open $\varepsilon$-balls with centers in all points of $X$ is called the **$\varepsilon$-neighbourhood** of $X$.

**Definition 4.10.** Let $M$ be a metric space and let $X$ and $Y$ be its bounded subsets. The **Hausdorff distance** $d_H(X, Y)$ is the infimum of all $\varepsilon$ such that $Y$ is contained in an $\varepsilon$-neighborhood of $X$ and $X$ is contained in an $\varepsilon$-neighborhood $Y$.

**Exercise 4.18 (!).** Prove that the Hausdorff distance defines a metric on the set $\mathcal{M}$ of all closed bounded subsets of $M$.

**Exercise 4.19.** Let $X$, $Y$ be bounded subsets of $M$ and $x \in X$. Prove that it is always the case that $d_H(X,Y) \geqslant d(x,Y)$.

**Exercise 4.20 (!).** Let $M$ be a complete metric space. Prove that $\mathcal{M}$ is also complete.

**Hint.** Consider a Cauchy sequence $\{X_i\}$ of subsets of $M$. Let $\mathfrak{S}$ be the set of Cauchy sequences $\{x_i\}$ with $x_i \in X_i$. Let $X$ be the set of accumulation points of sequences from $\mathfrak{S}$. Prove that $\{X_i\}$ converges to $X$.

**Exercise 4.21 (*).** Let $\{X_i\}$ be a Cauchy sequence of compact subsets of $M$ and $X$ be its limit. Prove that $X$ is compact.

**Hint.** One can identify $\{X_i\}$ with its subsequence such that

$$d_H(X_i, X_j) < 2^{-\min(i,j)}. \tag{4.1}$$

Consider a sequence $\{x_i\}$ of points from $X$. For every $X_j$ find a sequence $\{x_i(j) \in X_j\}$ such that $d(x_i(j), x_i) = d(x_i, X_j)$. Since $X_j$ is compact, this sequence has an accumulation point. Choose an accumulation point $x(0)$ in $\{x_i(0)\}$ and replace $\{x_i\}$ with its subsequence such that $\{x_i(0)\}$ converges to $x(0)$. Then replace $\{x_i\}, i > 0$ with a subsequence such that $\{x_i(1)\}$ converges to $x(1)$. We replace $\{x_i\}, i > k$ with a subsequence on $k$-the step in such a way that $\{x_i(k)\}$ converges to $x(k)$. Prove that we will finally obtain a sequence $\{x_i\}$ such that $\{x_i(k)\}$ converges to $x(k)$ for all $k$. Prove that this operation can be carried out in such a way that $d(x_i(k), x(k)) < 2^{-i}$. Use (4.1) to prove that $d(x_i(k), x_i) < 2^{-\min(k,j)+2}$. Deduce that $\{x_i\}$ is a Cauchy sequence.

**Exercise 4.22 (!).** Let $M$ be compact and $X \subset M$. Prove that for any $\varepsilon > 0$ there is a finite set $R \subset M$ such that $d_H(R, X) < \varepsilon$. (This statement can be rephrased as follows: "$X$ allows approximation by finite sets with any prescribed accuracy")

**Hint.** Find a finite $\varepsilon$-net in $X$.

**Exercise 4.23 (*).** Let $M$ be compact. Prove that $\mathcal{M}$ is also compact.

**Hint.** Use the previous problem.

**Definition 4.11.** Let $M$ be a metric space. It is said that $M$ is **locally compact**, if for any point $x \in M$ there exists a number $\varepsilon > 0$, such that the closed ball $\overline{B}_\varepsilon(x)$ is compact.

**Exercise 4.24.** Let $M$ be a locally compact metric space and $\overline{B}_\varepsilon(x)$ be a closed compact ball. Prove that $\overline{B}_\varepsilon(x)$ is contained in an open set $Z$ with compact closure.

**Hint.** Cover $\overline{B}_\varepsilon(x)$ with balls such that their closures are compact, and find a finite subcover.

**Exercise 4.25 (!).** Prove in the previous problem setting that for some $\varepsilon' > 0$ the ball $\overline{B}_{\varepsilon+\varepsilon'}(x)$ is also compact.

**Hint.** Take $Z$ as in the previous problem. Take $\varepsilon'$ to be $d(M\backslash Z, \overline{B}_\varepsilon(x))$.

**Definition 4.12.** Let $(M, d)$ be a metric space. It is said that $M$ **satisfies Hopf-Rinow condition** if for any two points $x, y \in M$ and for any two numbers $r_x, r_y > 0$ such that $r_x + r_y < d(x, y)$

$$d(B_{r_x}(x), B_{r_y}(y)) = d(x, y) - r_x - r_y.$$

**Exercise 4.26 (**).** If you know the definition of a Riemannian (or Finsler) manifold, prove that the Hopf-Rinow condition holds for the natural metric on such a manifold. Justify all the facts that you use in the proof.

**Exercise 4.27 (*).** Let $M$ be a complete locally compact metric space which satisfies Hopf-Rinow condition, $x \in M$ be a point and $\varepsilon > 0$ be a number such that $\overline{B}_{\varepsilon'}(x)$ is compact for all $\varepsilon' < \varepsilon$. Prove that the ball $\overline{B}_{\varepsilon}(x)$ is compact.

**Hint.** Let $\{\varepsilon_i\}$, with $\varepsilon_i < \varepsilon$, be a sequence that converges to $\varepsilon$. Use the Hopf-Rinow condition to prove that $\{\overline{B}_{\varepsilon_i}(x)\}$ is a Cauchy sequence with respect to Hausdorff metric, $\overline{B}_{\varepsilon}(x)$. Use the fact that the limit of such a sequence is compact (you have already proved it before).

**Exercise 4.28 (*).** (Hopf-Rinow theorem, I) Let $M$ be a complete locally compact metric space which satisfies Hopf-Rinow condition. Prove that every closed ball $\overline{B}_{\varepsilon}(x)$ in $M$ is compact.

**Exercise 4.29.** Let $M$ be a metric space such that every closed ball $\overline{B}_{\varepsilon}(x)$ in $M$ is compact. Prove that $M$ is complete.

**Exercise 4.30 (*).** Let $M$ be a locally compact complete metric space which satisfies Hopf-Rinow condition, $x, y \in M$. Prove that there is a point $z \in M$ such that $d(x, z) = d(y, z) = \frac{1}{2} d(x, y)$.

**Exercise 4.31 (*).** Let $S$ be a set of all rational numbers of the form $\frac{n}{2^k}$, $n \in \mathbb{Z}$ which belong to the interval $[0, 1]$. Prove in the previous problem setting that there exists a mapping $S \xrightarrow{\xi} M$ such that $d(\xi(a), \xi(b)) = |a - b| d(x, y)$ and $\xi(0) = x$ and $\xi(1) = y$.

**Exercise 4.32 (*).** (Hopf-Rinow theorem, II) Let $M$ be a locally compact complete metric space which satisfies Hopf-Rinow condition, $x, y \in M$. Prove that the mapping $\xi$ can be naturally extended to the completion of $S$ with respect to the standard metric, so that the resulting mapping $[0, 1] \xrightarrow{\overline{\xi}} M$ satisfies $\overline{\xi}(0) = x$, $\overline{\xi}(1) = y$ and $d((\overline{\xi}(a), \overline{\xi}(b)) = |a - b| d(x, y)$ for any two reals $a, b \in [0, 1]$.

**Remark.** Such a mapping $\overline{\xi}$ is called **geodesic**. The Hopf-Rinow theorem can be restated as follows: for any two points in a complete metric locally compact space which satisfies Hopf-Rinow condition there is a geodesic that connects them.

**Definition 4.13.** Such a space is called **geodesically connected**.

**Exercise 4.33 (*).** Give an example of a metric space, which is not locally compact but geodesically connected.

**Exercise 4.34.** Let $V = \mathbb{R}^n$ be the metric space with the standard (Euclidean) metric. Prove that geodesics in $V$ are intervals (sets of the form $ax + (1 - a)y$, where $a$ belongs to $[0, 1] \subset \mathbb{R}$, and $x, y \in V$).

**Exercise 4.35.** Let $V$ be a finite dimensional vector space with a norm that defines a metric $d$ and $d_0$ be the Euclidean metric on $V$. Prove that the identity mapping $(V, d) \to (V, d_0)$ is continuous iff a unit ball in $(V, d)$ contains a ball from $(V, d_0)$. Prove that the inverse mapping is continuous provided that a unit ball in $(V, d)$ is contained in a ball from $(V, d_0)$.

**Exercise 4.36.** In the previous problem settings, consider a function $D(x) := d(0, x)$ on a unit sphere $S^{n-1} \subset V$

$$S^{n-1} = \{x \in V \mid d_0(0, x) = 1\}$$

Let $D$ be a continuous function on $S^{n-1}$. Prove that the mapping $(V, d) \to (V, d_0)$ is continuous and the inverse mapping is continuous.

**Hint.** Use the fact that a continuous function on a compact set achieves its minimum and maximum values.

**Exercise 4.37 (**).** Prove that $D$ is a continuous function.

**Exercise 4.38.** Let $V$ be a finite dimensional vector space with a norm that defines the metric $d$. Suppose that the identity mapping $(V, d) \to (V, d_0)$ is continuous and the inverse mapping is also continuous. Prove that $(V, d)$ is complete and locally compact.

**Exercise 4.39 (*).** Let $d$ be the metric on $\mathbb{R}^n$ associated with the norm $(x_1, x_2, ...) \mapsto \max |x_i|$. Prove that it satisfies the Hopf-Rinow condition. Prove that $\mathbb{R}^n$ with such a metric is geodesically connected. Describe how the geodesics look like.

**Exercise 4.40 (*).** Is it true that the metric $d$ defined by a norm always satisfies the Hopf-Rinow condition?

**Definition 4.14.** Let $X$ be a metric space and $0 < k < 1$ be a real number. A mapping $f : X \to X$ is called **contraction mapping with a contraction coefficient** $k$ if $kd(x, y) \geqslant d(f(x), f(y))$.

**Exercise 4.41 (!).** Let $X$ be a meric space and $f : X \to X$ be a contraction mapping. Prove that for any $x \in X$ the sequence $\{a_i\}$, $a_0 := x, a_1 := f(x), a_2 := f(f(x)), a_3 := f(f(f(x))), \ldots$ is Cauchy sequence.

**Hint.** Use the fact that $d(a_i, a_{i+1}) = k^i d(x, f(x))$, and deduce that the series $\sum d(a_i, a_{i+1})$ converges.

**Exercise 4.42 (!).** (The Contraction Mapping Theorem) Let $X$ be a complete metric space and $f : X \to X$ be a contraction mapping. Prove that $f$ has a fixed point.

**Hint.** Find the limit of the sequence $x, f(x), f(f(x)), f(f(f(x))), \ldots$.

# GEOMETRY 5: Set-theoretic topology.

**Definition 5.1.** Consider a set $M$ and a collection of distinguished sets $S \subset \wp(M)$ called **open subsets**. A pair $(M, S)$ (and, by abuse of notation, $M$ itself) is called a **topological space**, if the following conditions are met:

1. An empty set and $M$ are open;
2. The union of any number of open sets is open;
3. The intersection of a finite number of open sets is open.

A mapping $\varphi: M \longrightarrow M'$ of topological spaces is called **continuous**, if the preimage of every open set is open. Continuous mappings are also called **morphisms** of topological spaces. An **isomorphism** of topological spaces is a morphism $\varphi: M \longrightarrow M'$ such that there is an inverse morphism $\psi: M' \longrightarrow M$ (i.e. $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity morphisms). An isomorphism of topological spaces is called **homeomorphism**.

A subset $Z \subset M$ is called **closed**, if its complement is open. A **neighborhood** of a point $x \in M$ is an open subset of $M$ which contains $x$. A **neighborhood** of a subset $Z \subset M$ is an open subset of $M$ that contains $Z$.

**Exercise 5.1.** Prove that a composition of continuous mappings is continuous.

**Exercise 5.2 (!).** Consider a set $M$ and let $S$ be a set of all subsets of $M$. Prove that $S$ defines a topology on $M$. This topology is called **discrete**. Describe a set of all continuous mappings from $M$ to a given topological space.

**Exercise 5.3 (!).** Consider a set $M$ and let $S$ be the set containing an empty set and $M$ itself. Prove that $S$ defines a topology on $M$. This topology is called **codiscrete**. Describe a set of all continuous mappings from $M$ to a space with discrete topology.

**Exercise 5.4.** Give an example of a continuous bijection between topological spaces that is not a homeomorphism.

**Exercise 5.5.** Consider a subset $Z$ of a topological space $M$. Open subsets of $Z$ are defined to be intersections of the form $Z \cap U$, where $U$ is open in $M$.

a. Prove that this defines a topology on $Z$. Prove that a natural embedding $Z \hookrightarrow M$ is continuous.

b. **(*)** Can all the continuous embeddings be obtained in this way?

**Definition 5.2.** Such a topology on $Z \subset M$ is said to be **induced by** $M$. We will consider any subset of any topological space as a topological space with induced topology.

**Definition 5.3.** Consider a topological space $M$, and let $S_0$ be such a collection of open sets such that any open set can be represented as a union of sets from $S_0$. Then $S_0$ is called a **base** of $M$.

**Exercise 5.6.** Describe all bases of a space $M$ with discrete topology; of a space $M$ with codiscrete topology.

**Definition 5.4.** Consider a metric space $M$. Recall that a subset $U \subset M$ is called **open**, if for every point $u \in U$, $U$ contains a ball of radius $\varepsilon > 0$ with the center $u$.

**Exercise 5.7.** Prove that this definition defines a topology on a metric space.

**Definition 5.5.** A topological space is called **metrizable** if it can be obtained from a metric space as described above.

**Exercise 5.8.** Prove that a discrete space is metrizable and a codiscrete space is not.

**Exercise 5.9.** Prove that open balls in a metric space $M$ are open. Prove that open balls define a base of topology on $M$.

**Exercise 5.10 (!).** Consider a topological space $M$ and two topologies $S$, $S'$ on $M$. Suppose that for every point $m \in M$ and every neighborhood $U' \ni m$ which is open in the topology $S'$ there is a neighborhood $U \ni m$, $U \subset U'$, which is open in the topology $S$. Prove that the identity mapping $(M, S) \xrightarrow{i} (M, S')$ is continuous. Give an example where $i$ is not a homeomorphism.

**Remark.** It is said in this case that the topology defined by $S'$ is **stronger** than the topology defined by $S$.

**Exercise 5.11.** Consider the space $\mathbb{R}^n$ with a norm $\nu$ (see GEOMETRY 3). This norm defines a metric and hence a topology on $\mathbb{R}^n$. Denote this topology by $S_\nu$. Let $\nu$, $\nu'$ be two norms satisfying $C^{-1}\nu'(x) < \nu(x) < C\nu'(x)$ for a fixed $C \in \mathbb{R}$. Prove that the identity mapping on $\mathbb{R}^n$ defines a homeomorphism $(\mathbb{R}^n, S_\nu) \longrightarrow (\mathbb{R}^n, S_{\nu'})$.

**Hint.** Use the previous problem.

**Exercise 5.12 (*).** Consider two norms $\nu$, $\nu'$ on $\mathbb{R}^n$ such that the identity mapping on $\mathbb{R}^n$ defines a homeomorphism $(\mathbb{R}^n, S_\nu) \longrightarrow (\mathbb{R}^n, S_{\nu'})$. Prove that there exists a constant $C$ such that $C^{-1}\nu'(x) < \nu(x) < C\nu'(x)$.

**Exercise 5.13 (*).** Consider a finite-dimensional vector space $V$ endowed with a symmetric positive bilinear form $g$. We will consider $V$ as a metric space with the metric $d_g$, constructed in GEOMETRY 3. Denote by $S_g$ the topology defined by $d_g$. Prove that the corresponding topology on $V$ does not depend upon $g$, i.e. for any (symmetric positive bilinear) $g, g'$, the identity map on $V$ is a homeomorphism $(V, S_g) \longrightarrow (V, S_{g'})$.

**Exercise 5.14 (**).** Consider a finite-dimensional vector space $V$ with norm $\nu$. Prove that the topology $S_\nu$ does not depend on norm $\nu$: the identity map on $\mathbb{R}^n$ is a homeomorphism $(\mathbb{R}^n, S_\nu) \longrightarrow (\mathbb{R}^n, S_{\nu'})$. Is it true for a infinite-dimensional $V$?

**Definition 5.6.** Consider a metric $d$ on $\mathbb{R}^n$, defined by the norm

$$|(\alpha_1, \dots, \alpha_n)| = \sqrt{\sum_i \alpha_i^2}.$$

The topology on $\mathbb{R}^n$, defined by $d$ is called the **natural** topology. The **natural topology** on subsets of $\mathbb{R}^n$ is the topology induced by the natural $\mathbb{R}^n$-topology.

**Exercise 5.15.** Consider $\mathbb{R}$ with the natural topology. Consider a space $M$ with discrete topology and a space $M'$ with a codiscrete topology. Find the set of all continuous maps

a. from $\mathbb{R}$ to $M$

b. from $M$ to $\mathbb{R}$

c. from $M'$ to $\mathbb{R}$

d. from $\mathbb{R}$ to $M'$.

**Exercise 5.16.** Consider a mapping $\varphi: M \longrightarrow M'$, where $M, M'$ are topological spaces. Is it true that the continuity of $\varphi$ implies that the preimage of any closed set is closed? Is it true that if a preimage of any closed set is closed then $\varphi$ is continuous?

**Exercise 5.17.** Give an example of a continuous mapping of topological spaces such that the image of an open set is not open. Give an example of a continuous mapping of topological spaces such that the image of a closed set is closed.

**Definition 5.7.** Consider a topological space $M$ and arbitrary $Z \subset M$. The intersection of the closed sets of $M$ containing $Z$ is denoted by $\overline{Z}$ and is called the **closure** of $Z$.

**Exercise 5.18.** Prove that $\overline{Z}$ is closed.

**Definition 5.8.** Consider a topological space $M$. The following conditions T0-T4 are called **separation axioms.**

**T0.** Let $x \neq y \in M$. Then at least one of the points $x$, $y$ has a neighborhood containing the other point.

**T1.** Every point in $M$ is closed.

**T2.** For any $x \neq y \in M$ there are non-intersecting neighborhoods $U_x$, $U_y$.

**T3.** For any point $y \in M$, every $M \supseteq U \ni y$ contains an open neighborhood $U' \ni y$ such that $U$ contains the closure of $U'$.

**T4.** For any closed subset $Z \in M$, any neighborhood $U \supset Z$ contains an open neighborhood $U' \supset Z$ such that $U$ contains the closure of $U'$.

The condition $T_2$ is widely known as the **Hausdorff axiom**. A topological space that satisfies the $T_2$ condition is called a **Hausdorff**.

**Exercise 5.19.** Prove that the condition $T_1$ is equivalent to the following one: for any two distinct points $x, y \in M$, there exists a neighborhood of $y$, which does not contain $x$.

**Exercise 5.20.** Prove that the condition $T_4$ is equivalent to the following one: any two distinct closed sets $X, Y \subset M$ have two non-intersecting neighborhoods.

**Exercise 5.21.** Let $M$ be a topological space. Consider an equivalence relation on $M$ defined the following way: $x$ is equivalent to $y$ iff $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Denote the set of equivalence classes as $M'$.

a. Verify that this is indeed an equivalence relation . Prove that $M$ satisfies the $T_0$ iff $M = M'$.

b. Define $U \subset M'$ to be open iff its preimage w.r.t. the mapping $M \to M'$ is open. Prove that this defines a topology on $M'$. Does it satisfy the $T_0$ condition?

c. Prove that the open subsets of $M$ are exactly the preimages of the open subsets of $M'$.

d. Suppose that $M$ has the codiscrete topology. What is $M'$?

**Exercise 5.22.** Are $T_0 - T_4$ conditions satisfied by a space with the discrete topology? With the codiscrete topology?

**Exercise 5.23.** Prove that $T_0 - T_4$ are satisfied by $\mathbb{R}$.

**Exercise 5.24.** Prove that $T_1$ implies $T_0$ and that $T_2$ implies $T_1$.

**Exercise 5.25.** Give an example of a space that does not satisfy the $T_1$ condition. Give an example of a non-Hausdorff space such that all the singleton sets are closed in it.

**Exercise 5.26 (*).** Give an example of a space that satisfies the $T_1$ condition such that any two non-empty open sets have an non-empty intersection.

**Exercise 5.27 (*).** Prove that $T_2$ follows from $T_1$ and $T_3$.

**Exercise 5.28 (*).** Give an example of a space that satisfies $T_4$ but does not satisfy $T_1$.

**Exercise 5.29.** Consider a metrizable topological space. Prove that it satisfies conditions $T_1$, $T_2$, $T_3$.

**Exercise 5.30 (*).** Consider a metrizable topological space. Prove that it satisfies the condition $T_4$.

**Exercise 5.31 (*).** Let $M$ be a finite set.

a. Find all topologies on $M$ that satisfy the $T_1$ condition.

b. Are there any topologies on $M$ that do not satisfy $T_1$?

c. Are there any topologies on $M$ that do not satisfy $T_1$, but satisfy $T_0$?

**Definition 5.9.** A set $M$ is said to be **partially ordered**, if there is a binary relation $x \leq y$ ("$x$ less than or equal $y$") defined on it such that:

1. If $x \leq y$ and $y \leq z$, then $x \leq z$.
2. If $x \leq y$ and $y \leq x$, then $x = y$.

**Exercise 5.32 (*).** a. Consider a partially ordered set $M$; say that $S \subset M$ is open if together with any $x \in S$ it contains all $y \in M$ satisfying $y \leq x$. Prove that this defines a topology on $M$. When does this topology satisfy the $T_0$ condition? The $T_1$ condition?

b. Consider a finite set $M$ and a topology on $M$ that satisfies the $T_0$ condition. Prove that it is induced by a partial order on $M$.

**Definition 5.10.** Let $Z \subset M$ be a subset of a topological space. A subset $Z$ is called **dense**, if $Z$ has a non-empty intersection with every open subset of $M$.

**Exercise 5.33 (!).** Prove that $Z$ is dense iff the closure $\overline{Z}$ is the entire $M$.

**Exercise 5.34.** Find all dense subsets in a topological space with the discrete topology; with the codiscrete topology.

**Exercise 5.35.** Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

**Exercise 5.36 (!).** A subset $Z$ in a topological space $M$ is called **nowhere dense**, if for every open $U \subset M$ the subset $Z \cap U$ is not dense in $U$. Prove that $Z$ is nowhere dense iff $M \backslash \overline{Z}$ is dense in $M$.

**Exercise 5.37 (*).** Construct a nowhere dense subset of the interval $[0, 1]$ (endowed with the natural topology) of the continuum cardinality.

**Exercise 5.38.** Find all nowhere dense subsets in a space with discrete topology; with codiscrete topology.

**Definition 5.11.** Let $M$ be a topological space and $x \in M$ be an arbitrary point. A neighborhood base of $x$ is a collection $B$ of neighborhoods of $x$ such that any neighborhood $U \ni x$ contains some neighborhood from $B$.

**Exercise 5.39.** Consider a collection $B$ of open subsets of a topological space $M$ such that for any $x \in M$ the collection of all $U \in B$ containing $x$ is a neighborhood base of $x$. Prove that $B$ is a base of the topology of $M$.

**Definition 5.12.** Consider a topological space $M$. One can impose two countability conditions on $M$. If every point of $M$ has a countable neighborhood base, then it is said that $M$ satisfies **the first countability axiom**. If $M$ has a countable base of open sets, then it is said that $M$ satisfies **the second countability axiom**, or that $M$ is a **space with a countable base**. If there exists a countable dense subset of $M$ then it is said that $M$ is **separable**.

**Exercise 5.40.** Consider a space $M$ with discrete topology. Prove that $M$ satisfies the first countability axiom.

**Exercise 5.41.** Consider a topological space $M$ with a countable base. Prove that it is separable.

**Exercise 5.42 (*).** Consider a separable topological space $M$. Prove that $M$ has a countable base.

**Exercise 5.43 (!).** Consider a metrizable topological space. Prove that it has a countable neighborhood base for every point.

**Exercise 5.44.** Construct a non-separable metrizable topological space.

**Exercise 5.45 (**).** Give an example of a countable Hausdorff space without a countable base.

## 5.1 Topology and convergence

Topological space were invented as a language to speak about continuous functions. In GEOMETRY 4 we defined a continuous function as a function that preserves limits of convergent sequences. One can consider topology from the axiomatic viewpoint as above, or from the point of view of geometric intuition, by giving a class of convergent sequences on a space to define its topology and considering a mapping continuous if it preserves limits.

The second approach (despite all its obvious advantages) encounters set-theoretical problems: if the space does not have a countable base, then one has to use well-founded uncountable sequences. We are going to work mostly with spaces which have a countable neighborhood base and it is convenient to define topology and continuity via limits of sequences.

**Definition 5.13.** Let $M$ be a topological space and $Z \subset M$ be an infinite subset. A point $x \in M$ is called an **accumulation point** of $Z$, if every neighborhood of $x$ contains some point $z \in Z$. A **limit** of a sequence $\{x_i\}$ is defined to be a point $x$ such that every neighborhood of $x$ contains almost all $x_i$'s. A sequence is called **convergent** if it has a limit.

**Exercise 5.46.** Find all convergent subsequences in a space with discrete topology; in a space with codiscrete topology.

**Exercise 5.47.** Consider a Hausdorff space $M$. Prove that every sequence has at most one limit.

**Exercise 5.48 (*).** Is the converse true (i.e. does it follow from the uniqueness of a limit that the space is Hausdorff)? What if $M$ has a countable neighborhood base of its point?

**Exercise 5.49.** Consider a space $M$ where any sequence has at most one limit. Prove that $M$ satisfies the separation axiom $T_1$.

**Exercise 5.50.** Consider a continuous mapping $f: M \longrightarrow M'$ and a subset $Z \subset M$. Prove that $f$ maps accumulation points of $Z$ to accumulation point of $f(Z)$. Prove that $f$ maps limits to limits.

**Exercise 5.51 (!).** Consider a mapping that maps accumulation points to accumulation points. Prove that it is continuous.

**Exercise 5.52.** Consider a space $M$ with a countable neighborhood base for every point, and an arbitrary $Z \subset M$. Prove that the closure of $Z$ is the set of limits of all sequences from $Z$.

**Exercise 5.53 (!).** Consider topological spaces $M$, $M'$ with a countable neighborhood base for every point and a mapping $f: M \longrightarrow M'$ that preserves limits of sequences. Prove that $f$ is continuous.

**Hint.** Use the previous problem.

**Exercise 5.54 (*).** What happens if we do not require in the previous problem that neighborhood bases are countable in $M$? In $M'$?

**Exercise 5.55 (*).** Consider a set $M$ and let some sequences of elements of $M$ be declared to **converge** to points from $M$ (it is denoted like this: $x \in \lim x_i$; note that there can be more than one limit of a sequence[1]). Let the following conditions hold for the notion of convergence:

(i) The limit of a sequence $x, x, x, x, x, \ldots$ contains $x$.

(ii) If $x \in \lim x_i$ then the limit of any subsequence $\{x_{i_\ell}\}$ is nonempty and contains $x$.

(iii) Consider an infinite number of elements of a sequence $\{x_i\}$. Let us permute them and denote the result by $\{y_i\}$. If $x \in \lim x_i$ then $x \in \lim y_i$.

(iv) If $x \in \lim x_i$ and $x \in \lim y_i$ then the sequence $x_1, y_1, x_2, y_2, ...$ converges to $x$.

a. Define closed subsets of $M$ as these $Z \subset M$ that contain the limits of all sequences $\{x_i\} \subseteq Z$. Define open sets as complements of closed sets. Prove that this defines a topology on $M$.

[1] Thus we talk here about limits of sequences as *sets* of points. (DP)

b. Consider a topology $S$ on $M$ with a countable neighborhood base for every point. Define the limits of sequences with respect to this topology. Prove that conditions (i)-(iv) hold for this notion of convergence. Let $S'$ be a topology obtained from limits with the help of construction in (a). Prove that the topologies $S'$ and $S$ coincide.

c. Take a uncountable set with the following topology: open sets are complements of finite sets (this topology is called cofinite). Consider a topology $S'$ defined by limits as above. Describe $S'$. Prove that $S'$ does not satisfy the first countability axiom.

# GEOMETRY 6: Set-theoretic topology: product of spaces

**Definition 6.1.** Consider a topological space $M$ and a collection $B$ of open subsets of $M$. The collection $B$ is called a **prebase** of the topology on $M$, if every open set can be obtained as a union (potentially infinite) of finite intersections of open subsets from $B$.

**Exercise 6.1.** Consider $\mathbb{R}$ with discrete topology. Prove that it does not have a countable prebase.

**Exercise 6.2 (!).** Consider a topological space $M$ with a countable prebase. Prove that $M$ has a countable base.

**Exercise 6.3 (*).** Consider a finite set $M$, $|M| = 2^n$ with discrete topology and a prebase $B$ of $M$. Prove that $|B| \geqslant 2n$. Find a prebase that has $2n$ elements.

**Exercise 6.4.** Consider $\mathbb{R}$ with natural topology and let $B$ be the set of all intervals such that their end-points are finite binary fractions. Prove that $B$ is a base of topology of $\mathbb{R}$.

**Exercise 6.5.** Consider a collection $B$ of subsets of a set $M$ such that $\cup B = M$. Consider all subsets of $M$ that are finite intersections and arbitrary unions of elements of $B$, as well as $M$ and $\emptyset$. Prove that these sets define a topology on $M$.

**Definition 6.2.** This topology is called the **topology defined by the prebase** $B$.

**Definition 6.3.** Consider topological spaces $M_1$ and $M_2$. Consider a topology $S$ on $M_1 \times M_2$ defined by the prebase of subsets of the form $U_1 \times M_2$, $M_1 \times U_2$ where $U_1$, $U_2$ are open in $M_1$, $M_2$ respectively. Then $(M_1 \times M_2, S)$ is called the **product of $M_1$ and $M_2$**.

**Exercise 6.6.** Prove that the natural projection $M_1 \times M_2 \longrightarrow M_1$ is continuous. Prove that sets of the form $U_1 \times U_2$ define a base of the topology of $M_1 \times M_2$.

**Exercise 6.7.** Consider mappings of topological space $X \xrightarrow{\gamma_1} M_1$, $X \xrightarrow{\gamma_2} M_2$. Prove that they are continuous iff the product

$$X \xrightarrow{\gamma_1 \times \gamma_2} M_1 \times M_2$$

is continuous.

**Exercise 6.8.** Consider topological spaces $M_1$, $M_2$ that have one of the properties from the list below. Prove that $M_1 \times M_2$ has the same property.

a. Separation axiom $T_1$.

b. **(!)** Hausdorff separation axiom ($T_2$).

c. Separation axiom $T_3$.

d. Being separable.

e. **(!)** Having a countable neighborhood base for every point.

f. Having a countable base.

**Exercise 6.9 (**).** Does this hold for the separation axiom $T_4$? What about $T_4 + T_1$?

**Definition 6.4.** The mapping $x \xrightarrow{\Delta} (x, x) \in X \times X$ is called the **diagonal embedding** and its image is called the **diagonal** in $X \times X$.

**Exercise 6.10.** Prove that the diagonal embedding is a homeomorphism onto its image (supposing that the topology on $\Delta \subset X \times X$ is induced from $X \times X$).

**Hint.** Use the Problem 6.7.

**Exercise 6.11.** Prove that $X$ satisfies the $T_1$ separation axiom iff the diagonal is the intersection of all open sets that contain it.

**Exercise 6.12 (!).** Prove that $X$ is Hausdorff iff the diagonal is closed in $X \times X$.

**Exercise 6.13 (*).** Suppose that the graph $\Gamma \subset X \times Y$ of a mapping of topological spaces $X \xrightarrow{\gamma} Y$ is closed. Is it true that $\gamma$ is continuous?

**Exercise 6.14 (!).** Consider a morphism of topological spaces $X \xrightarrow{\gamma} Y$ and suppose $X$ is Hausdorff. Prove that the graph of $\gamma$ is closed.

**Exercise 6.15.** Consider metric spaces $M_1$, $M_2$ and their product $M = M_1 \times M_2$, and let $d$ be one of the functions defined on $M \times M$ listed below. Prove that $d$ defines a metric on $M$.

a. $d((m_1, m_2), (m_1', m_2')) = d(m_1, m_2) + d(m_1', m_2')$

b. $d((m_1, m_2), (m_1', m_2')) = \max(d(m_1, m_2), d(m_1', m_2'))$

c. **(!)** $d((m_1, m_2), (m_1', m_2')) = \sqrt{d(m_1, m_2)^2 + d(m_1', m_2')^2}$

**Exercise 6.16 (!).** Prove that all the three metric structures from the previous problem define the same topology on $M_1 \times M_2$. Prove that this topology is equivalent to the topology of the product $M_1 \times M_2$ considered as a product of topological spaces.

## Tychonoff cube and Hilbert cube

**Definition 6.5.** Consider a (possibly uncountable) index set $I$ and the set $M = X^I$ of all mappings from $I$ to a fixed topological space $X$. One can regard $X^I$ as a set of sequences of points of $X$ indexed by $I$ or as an infinite product of $X$ with itself. Denote by $W(i, U) \subset X^I$ the set of all mappings $I \longrightarrow X$ that map a fixed index $i$ to an element from a subset $U \subset X$. Define a prebase $B$ of topology on $X^I$ in the following way: let $V \in B$ if $V = W(i, U)$ for some index element $i \in I$ and some open subset $U \subset X$. This topology is called **weak**.

**Exercise 6.17 (!).** Consider a sequence of points $\alpha_1, \alpha_2, \dots$ in $X^I$. Prove that it converges iff the sequence $\alpha_k(i)$ converges for every index $i \in I$.

**Remark.** The previous problem statement is often expressed as follows: "a space $X^I$ with weak topology is the set of mappings from $I$ to $X$ with the pointwise convergence topology".

**Definition 6.6.** Consder an index set $I$. The space $[0, 1]^I$ with the weak topology is called a **Tychonoff cube**.

**Exercise 6.18.** Consider a set of continuous functions $\alpha_i : M \longrightarrow [0, 1]$ indexed by a set $I$. Prove that the mapping of the form

$$\prod \alpha_i : m \longrightarrow \prod_{i \in I} \alpha_i(m)$$

from $M$ to Tychonoff cube $[0, 1]^I$ is continuous.

**Exercise 6.19.** Prove that any point of a Tychonoff cube is closed.

**Exercise 6.20 (*).** Prove that a Tychonoff cube satisfies $T_2$ and $T_3$ separation axioms.

**Exercise 6.21 (!).** Consider a Tychonoff cube $[0,1]^I$ where $I$ is countable. Prove that it has a countable base.

**Hint.** Prove that the collection of all $U = W(i, ]a,b[)$ with $a, b$ rational numbers defines a countable prebase in $[0,1]^I$ and use the Problem 6.2.

**Exercise 6.22 (**).** Prove that if the index set $I$ has the cardinality greater than or equal to continuum then the Tychonoff cube $[0,1]^I$ is non-separable.

**Hint.** Consider a countable subset $W$ of a Hausdorff space. Prove that the cardinality of the closure of $W$ is not greater than continuum.

**Exercise 6.23 (!).** Consider a set $M = [0,1]^{\mathbb{N}}$ of sequences of real numbers in $[0,1]$ indexed by $\mathbb{N}$. Consider the function $d: M \times M \longrightarrow \mathbb{R}$,

$$d(\{\alpha_i\}, \{\beta_i\}) = \sqrt{\sum_i i^{-2} |\alpha_i - \beta_i|^2}.$$

Prove that this function is well-defined and defines a metric on $[0,1]^{\mathbb{N}}$.

**Definition 6.7.** A metric space $[0,1]^{\mathbb{N}}$ with the metric defined as above is called **Hilbert cube**.

**Exercise 6.24 (!).** Consider a sequence $\{\alpha_i(n)\}$ of points of $[0,1]^{\mathbb{N}}$. Prove that it converges in the Tychonoff topology iff it converges in the topology of the Hilbert cube.

**Exercise 6.25 (*).** Deduce that the identity mapping is a homeomorphism of the Tychonoff cube and the Hilbert cube.

**Remark.** We actually proved that if the index set $I$ is countable then the Tychonoff cube $[0,1]^I$ is metrizable.

**Exercise 6.26 (*).** Consider a uncountable index set $I$. Is the Tychonoff cube $[0,1]^I$ metrizable in that case?

## Urysohn lemma and metrization of topological spaces

**Definition 6.8.** Consider two non-intersecting closed subsets $A, B \subset M$ of a topological space $M$. A continuous function $f: M \longrightarrow [0,1]$ is called an **Urysohn function** if $f(A) = 0, f(B) = 1$.

**Exercise 6.27.** Suppose that Urysohn function exists for any two non-intersecting closed subsets $A, B \subset M$. Prove that $M$ satisfies the separation axiom $T_4$.

**Exercise 6.28.** Prove in the previous problem setting that it is possible that $M$ does not satisfy $T_1$ separation axiom.

**Exercise 6.29 (*).** Suppose $M$ satisfies $T_4$ separation axiom and $A, B \subset M$ are non-intersecting and closed. Prove that there exists a sequence of neighborhoods $U_{p/2^q} \supset A$ indexed by rational numbers of the form $0 < p/2^q < 1$ that satisfies the following conditions:

(i) for all $p, q$, $B$ does not intersect $U_{p/2^q}$.

(ii) if $p_1/2^{q_1} < p_2/2^{q_2}$ then the closure of $U_{p_1/2^{q_1}}$ is contained in $U_{p_2/2^{q_2}}$.

**Hint.** Use an inductive argument.

**Exercise 6.30 (*).** In the previous problem setting define a function $f : M \longrightarrow [0,1]$ to be

$$f(m) = \sup \left\{ p_2/2^{q_2} \mid m \notin U_{p_1/2^{q_1}} \right\}$$

outside $A$ and equal to zero on $A$. Prove that $f$ is continuous and that $f$ is an Urysohn function.

**Hint.** Prove that the intervals of the form $]p_1/2^{q_1}, p_2/2^{q_2}[$ form a prebase of the topology on $[0,1]$. Prove that

$$f^{-1}(]p_1/2^{q_1}, p_2/2^{q_2}[) = U_{p_2/2^{q_2}} \backslash \overline{U_{p_1/2^{q_1}}}.$$

Deduce that $f$ is continuous.

**Remark.** We have proven the following "Urysohn lemma": if $M$ satisfies the $T_4$ condition, then for any two non-intersecting closed subsets of $M$ there is an Urysohn function.

**Exercise 6.31 (*).** Consider a Hausdorff space $M$ with a countable base $B$, which satisfies the $T_4$ condition and let $I$ be a set of all pairs $U_1, U_2 \in B$ such that the closures of $U_1$, $U_2$ do not intersect, $F_{U_1,U_2}$ are respective Urysohn functions and $F : M \longrightarrow [0,1]^I$ is a mapping to Tychonoff cube define as $F(m) = \prod F_{U_1,U_2}$. Prove that $F$ is continuous and injective.

**Exercise 6.32 (*).** In the previous problem setting denote the inverse mapping of $F$ as $G : F(M) \longrightarrow M$. Consider a sequence of points $\{x_i\}$ such that $F_{U_1,U_2}(x_i)$ converges for every pair $(U_1, U_2)$ from $I$. Deduce that the sequence $\{x_i\}$ converges. Prove that $G$ is continuous.

**Exercise 6.33 (*).** Prove that any Hausdorff space $M$ with a countable base which satisfies the $T_4$ condition (such space is called a **Polish** space) is a subspace of a Hilbert cube.

**Remark.** We have proved the following **metrization theorem**: every Polish space is metrizable.

**Exercise 6.34.** Prove that any subset of a Hilbert cube is Polish.

**Exercise 6.35 (*).** Is it true that every metrizable space is a Polish space?

# GEOMETRY 7: Set-theoretic topology: compactness

**Definition 7.1.** Consider a topological space $M$. We call any collection of open subsets $U_i \subset M$ (possibly infinite or even uncountable) such that $M = \bigcup U_i$ a **cover** of $M$ . The topological space $M$ is called **compact** (or **a compactum**) if it is possible to find a finite subcover of every open cover of $M$. A subset $Z \subset M$ of the topological space $M$ is called compact if it is compact in the induced topology.

**Exercise 7.1.** Prove that the interval $[0, 1]$ is compact. In which case a set with discrete topology is compact? With codiscrete topology?

**Exercise 7.2 (*).** Consider the following topology on $M$: open sets are complements of finite sets (this topology is called **cofinite**). Find all compact subsets of $M$.

**Exercise 7.3 (!).** Consider a compact space $Z$ and a closed subset $Z' \subset Z$. Prove that $Z'$ is also compact. Does compactness of a set follow from its closedness?

**Exercise 7.4.** Consider a Hausdorff topological space $M$, an arbitrary subset $Z$ of $M$ and a point $x \notin Z$.

a. Prove that there is an open cover $\{U_i\}$ of $Z$ such that the closure of every $U_i$ does not contain $x$.

b. **(*)**Give an example of a non-Hausdorff $T_1$-space where this is not true.

**Exercise 7.5 (!).** Consider a Hausdorff space $M$. Prove that every compact subset of $M$ is closed.

**Hint.** Use the previous problem.

**Exercise 7.6.** Consider two compact subsets of a Hausdorff space. Prove that there exist two non-intersecting open neighborhoods of these subsets.

**Exercise 7.7 (!).** Consider a compact Hausdorff topological space. Prove that it satisfies the $T_4$ separation axiom.

**Definition 7.2.** A topological space is called **locally compact** if every point has a neighborhood such that its closure is compact.

**Exercise 7.8.** Consider a locally compact Hausdorff topological space. Prove that is satisfies the $T_3$ separation axiom.

**Exercise 7.9 (**).** Does there exist a locally compact topological space which does not satisfy the first countability axiom?

**Exercise 7.10 (**).** Does there exist a countable topological space which is not locally compact?

**Exercise 7.11.** Consider a topological space $X$. Denote by $\widehat{X}$ the set $X \bigcup \{\infty\}$ ($X$ with one point added, this point is denoted by $\infty$) with the following topology: $U \subset \widehat{X}$ is open if either $\infty \in U$ and the complement of $U$ is compact as a subset of $X$, or if $\infty \notin X$ and $U$ is open as a subset of $X$. Prove that this is indeed a topology and that the space $\widehat{X}$ is compact.

**Definition 7.3.** The space $\widehat{X}$ is called a **one-point compactification** of the space $X$.

**Exercise 7.12 (*).** Consider a Hausdorff space $X$. Is it true that $\widehat{X}$ is also Hausdorff?

**Exercise 7.13.** Consider the space $X = \mathbb{R}^n$ with the natural topology. Prove that $\widehat{X}$ is homeomorphic to the $n$-dimensional sphere.

**Exercise 7.14.** Consider a topological space $M$ and a subset $Z$. Prove that the following are equivalent:

(i) Every point $z \in Z$ has a neighborhood $U \ni z$ that contains no other points from $Z$.

(ii) $M$ induces the discrete topology on $Z$.

(iii) $Z$ does not contain any of its accumulation points.

**Definition 7.4.** A closed subset $Z \subset M$ that satisfies one of the conditions from the statement of Problem 7.14 is called **discrete**.

**Exercise 7.15.** Consider a Hausdorff space $M$ and suppose it has an infinite discrete subset $Z \subset M$. Prove that $M$ is non-compact.

Consider a collection $Z_i$ of subsets of a set $M$. We say that the collection is **incomplete**, if for every finite subcollection $Z_1, Z_2, \dots, Z_k$ the intersection $Z_1 \cap Z_2 \cap \dots \cap Z_k$ is non-empty. A **monotone collection** $Z_i$ of subsets of the set $M$ is a collection of subsets that is linearly ordered by inclusion (i.e. for all $Z_i$, $Z_j$ from the collection either $Z_i \subset Z_j$, or $Z_j \subset Z_i$).

**Exercise 7.16.** Prove that a topological space $M$ is compact iff every incomplete collection of closed subsets $Z_i \subset M$ has a non-empty intersection $\cap_i Z_i$.

**Exercise 7.17.** Prove that if a topological space $M$ is compact then every monotone collection of non-empty closed subsets $Z_i \subset M$ has a non-empty intersection $\cap_i Z_i$.

**Exercise 7.18 (!).** Consider a Hausdorff topological space $M$ with a countable base. Prove that $M$ is compact iff $M$ does not have infinite discrete subsets.

**Hint.** If $M$ has an infinite discrete subset then it follows from the Problem 7.17 that $M$ is non-compact. Conversely, if $M$ is non-compact then $M$ has a countable cover $S$ such that no finite subset of $S$ covers $M$.

**Exercise 7.19.** Consider a Hausdorff topological space $M$ with a countable base. Prove that $M$ is compact iff every sequence of points from $M$ has an accumulation point.

**Exercise 7.20 (*).** Consider a topological space $M$, not necessarily Hausdorff.

a. Is it possible that a compact subset of $M$ contains an infinite discrete subset?

b. Is it possible that there is a non-compact subset of $M$ that contains no infinite discrete subsets?

c. **(**)** Consider a Hausdorff space $M$. Does there exist a non-compact subset of $M$ that does not contain infinite discrete subsets?

**Exercise 7.21 (!).** Consider a continuous mapping $f: M \longrightarrow N$ of topological spaces. Prove that for any compact subset $Z \subset M$, $f(Z)$ is compact.

**Exercise 7.22.** Consider a subset $Z \subset \mathbb{R}$.

a. Prove that $Z$ is compact iff it is closed and bounded (i.e. contained in an interval $[a, b]$).

b. Prove that $Z$ is compact iff every subset of it has an infimum and supremum in $Z$.

**Exercise 7.23 (!).** Consider a continuous mapping $f: M \longrightarrow \mathbb{R}$ of topological spaces. Prove that $f$ reaches its maximum and minimum on any compact subset of $M$.

**Exercise 7.24 (*).** Consider a non-compact Hausdorff topological space with a countable base that satisfies the $T_4$ separation axiom. Construct a continuous function $f: M \longrightarrow \mathbb{R}$ that has no maximum.

**Hint.** Consider $\{x_i\}$, a countable discrete subset of $M$. Use the $T_4$ separation axiom to construct a collection of neighborhoods $U_i \supset x_i$ such that the closure of $U_i$ does not intersect with the closure of $\bigcup_{j \neq i} U_j$. Now apply Urysohn lemma to closed sets $\{x_i\}$, $M \backslash U_i$ and sum up the Urysohn functions $f_i$ obtained with the right coefficients.

**Exercise 7.25.** Consider a continuous mapping $f: M \longrightarrow N$ of topological spaces, where $M$ is compact and $N$ is Hausdorff. Prove that $f$ maps closed sets to closed sets.

**Exercise 7.26.** Consider a continuous mapping $f: M \longrightarrow N$ of topological spaces, where $M$ is compact and $N$ is Hausdorff. Suppose that $f$ is bijective. Prove that $f$ is a homeomorphism.

**Exercise 7.27.** Give an example of a continuous mapping $f: M \longrightarrow N$, where $M$ is compact, such that $f$ is not a homeomorphism ($N$ is not Hausdorff here).

## Compact sets and products

**Definition 7.5.** A continuous mapping $f: X \to Y$ of topological spaces is called **proper** if for every compact $K \subset Y$ the preimage $f^{-1}(K) \subset X$ is compact.

**Exercise 7.28 (!).** Consider a Hausdorff space $Y$ with a countable base. Prove that a proper mapping $f: X \to Y$ maps closed subsets of $X$ to closed subsets of $Y$.

**Hint.** Take a closed set $Z \subset Y$ which has a non-closed image. Take a sequence of points $y_i \in f(Z)$ which converges to $y \in Y$ that does not belong to $f(Z)$.

**Exercise 7.29 (*).** Is the previous problem statement true if we do not require existence of a countable base?

**Exercise 7.30 (*).** Consider a continuous mapping $f: X \to Y$ that maps closed sets to closed sets and the preimage $f^{-1}(y)$ of any point $y \in Y$ is compact. Prove that the mapping $f$ is proper.

**Hint.** Use the compactness criterion from the Problem 7.16.

**Definition 7.6.** A continuous mapping $f: X \to Y$ is called **closed** if the image of any any closed subset is closed. The mapping is called **universally closed** if for any continuous mapping $g: Z \to Y$ the induced mapping $X \times_Y Z \to Z$ is closed ($X \times_Y Z$ is a subset of $X \times Z$ that contains all pairs $\langle x, z \rangle$ such that $f(x) = g(z)$).

**Exercise 7.31 (*).** Consider a continuous mapping $F : X \to Y$ which is universally closed. Prove that it is a proper mapping.

**Hint.** Use the Problem 7.30 to justify that only the case when $Y$ is a one point space can be considered. Then use the Problem 7.18: if $X$ contains an infinite discrete subset $M$ then take $Z = \widehat{M}$, i.e. a one-point compactification of $M$ and deduce the contradiction.

**Exercise 7.32 (!).** Consider compact topological spaces $X, Y$. Prove that the product $X \times Y$ is compact.

**Hint.** Use the fact that sets of the form $U \times V$, where $U$ is open in $X$ and $V$ is open in $Y$, form a base of the topology on $X \times Y$ and prove that it suffices to consider covers of $X \times Y$ that contain only sets of this form. Then for every point $y \in Y$ choose a finite subcover of the subset $X \times \{y\} \subset X \times Y$ that contains sets of the form $U_i \times V_i$, and notice that sets $V_y = \cap V_i$ form an open cover of $Y$.

Thus every projection $X \times Y \to Y$ for any $Y$ and compact $X$ is a proper mapping.

**Exercise 7.33.** Consider a subset $X \subset \mathbb{R}^n$. Prove that the following are equivalent:

(i) $X$ is compact

(ii) $X$ is closed and bounded (i.e. lies within a ball).

## Tychonoff's theorem

**Exercise 7.34.** Consider a sequence $a_i(n)$ of mappings from $\mathbb{N}$ to $[0, 1]$. Prove that one can select a subsequence $a_{i_1}, a_{i_2}, a_{i_3}, \ldots$ such that $\{a_{i_k}(n)\}$ converges for any $n$.

**Exercise 7.35 (!).** Deduce that the Tychonoff cube $[0, 1]^{\mathbb{N}}$ is compact.

**Exercise 7.36 (*).** Consider a topological space $M$. Consider a (possibly uncountable) collection $\{V_\alpha\}$ of covers of $M$, such that every $V_\alpha$ either contains $V_{\alpha'}$ or is contained in it (in other words, in $\{V_\alpha\}$ every cover can be obtained from any other cover by adding or removing some elements). Suppose every $V_\alpha$ does not have a finite subcover. Prove that the union of all $V_\alpha$ does not have a finite subcover either.

**Exercise 7.37 (*).** Use the Zorn's lemma to prove that every non-compact subset $X \subset M$ has a cover $\{V_\alpha\}$ that does not have a finite subcover, but if one adds to $\{V_\alpha\}$ any open set that does not belong to it, then the cover obtained has a finite subcover.

**Hint.** Use the previous problem.

We will call such covers **maximal**.

**Exercise 7.38 (*).** Consider a maximal cover $\{V_\alpha\}$ of a non-compact topological space $M$. Prove that if open sets $U_1$, $U_2$ do not belong to $\{V_\alpha\}$ and they have a non-empty intersection then the intersection does not belong to $\{V_\alpha\}$ either. Prove that any non-empty finite intersection of open sets that do not belong to $\{V_\alpha\}$, does not belong to $\{V_\alpha\}$ either.

**Hint.** Use the previous problem.

**Exercise 7.39 (*).** Consider a topological space $M$ with a given prebase of topology $R$. Consider then a non-compact subset $X \subset M$ and a maximal cover $\{V_\alpha\}$. Prove that $\{V_\alpha\}$ has a subcover whose elements belong to $R$.

**Hint.** Use the previous problem.

**Remark.** We have proved the following theorem (Alexander's theorem about prebase). Consider a topological space $M$ with a given prebase $S$. Then a subset $X \subset M$ is compact iff every cover of $X$ whose elements are from $S$ has a finite subcover. Alexander's theorem uses the Axiom of Choice and is equivalent to it (that was shown by Cayley).

**Exercise 7.40 (*).** Deduce that the Tychonoff cube $[0,1]^I$ is compact for any index set $I$.

**Hint.** Consider a prebase of the topology on the Tychonoff cube that consists of subsets of the form $[0,1]\times[0,1]\times\dots\times]a,b[\times[0,1]\times\dots$ (an open interval occurs once). Use Alexander's theorem.

**Remark.** Compactness of the Tychonoff cube is equivalent to the following statement. Consider a space $\mathrm{Map}(I,[0,1])$ of mappings from a set $I$ to the interval $[0,1]$, endowed with the topology of the pointwise convergence. Then $\mathrm{Map}(I,[0,1])$ is compact. In particular, every sequence $\{a_i(x)\}$ of mappings has a subsequence $\{a_{i_k}(x)\}$ such that $\{a_{i_k}(x)\}$ converges for all $x \in I$.

**Definition 7.7.** Consider a topological space $M$, a set $I$ and $M^I$, the set of all mappings from $I$ to $M$, that is, the product of $I$ copies of $M$. For an arbitrary $x \in I$ and an open set $U \subset M$ consider a subset $U(x) \subset M^I$ which consists of all mappings that map $x$ to an element of $U$. Define a topology on $M^I$ using the prebase that consists of all $U(x)$. This topology is called **Tychonoff topology** (or **weak topology** or **topology of pointwise convergence**).

**Exercise 7.41 (*).** Consider a compact space $M$. Deduce from Alexander's theorem that $M^I$ endowed with Tychonoff topology is compact.

## Fundamental theorem of algebra

Consider a polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ of a positive degree with complex coefficients. We look at $P$ as a function from $\mathbb{C}$ to $\mathbb{C}$. $\mathbb{C}$ is identified with $\mathbb{R}^2$ as a topological space.

**Exercise 7.42.** Prove that $P$ is continuous.

**Exercise 7.43 (!).** Prove that if $|x| > 2\max(1, \sum |a_i|)$, then $\frac{|P(x)-x^n|}{|x^n|} < 1/2$.

**Exercise 7.44 (!).** Prove that if $|x| > 2R\max(1, \sum |a_i|)$, then $|P(x)| > R^n$.

**Exercise 7.45 (!).** Deduce that $|P|$ reaches its local minimum at a point $a \in \mathbb{C}$.

**Hint.** We approximated the polynomial $|P|$ with the polynomial $x^n$, for which we know how fast it grows. We deduce that $|P(x)| > R^n$, when $|x|$ is big enough. That's why the minimum of $|P|$ on the disc $|x| \leqslant R$ is reached inside the disc and not on its boundary.

In order to simplify the notation we will suppose that $|P|$ reaches its minimum at zero. We want to prove that the minimum of $|P|$ is zero. Suppose it is not true. Then let $k$ be the smallest number among $1,2,3,\dots,n$, such that $a_k \neq 0$. Multiply $P$ by $a_0^{-1}$ and perform the substitution $x = z\sqrt[k]{a_k^{-1}}$, so we get a polynomial of the form

$$Q(z) = 1 + z^k + b_{k+1}z^{k+1} + b_{k+2}z^{k+2} + \dots$$

**Exercise 7.46.** Prove that for any complex $z$, such that $|z| < 1$, the following holds:

$$|Q(z) - 1 - z^k| < |z^{k+1}|(\sum |b_i|).$$

**Exercise 7.47 (!).** Prove that for any complex number $z$, such that $|z| < \frac{1}{2} \max\left(1, \sum |b_i|\right)^{-1}$, the following holds:

$$\frac{|Q(z) - 1 - z^k|}{|z^k|} < \frac{1}{2}.$$

**Exercise 7.48 (!).** Deduce that for any positive real $\varepsilon < \frac{1}{2} \max\left(1, \sum |b_i|\right)^{-1}$ and any complex $z$, such that $z^k = -\varepsilon$, the following holds:

$$|Q(z) - 1 + \varepsilon| < \varepsilon/2.$$

**Remark.** We approximated $Q$ with the polynomial $1 - z^k$ in a neighbourhood of zero. We can use this approximation to deduce that $|Q(\sqrt[k]{-\varepsilon})| < |Q(0)|(1 - \frac{1}{2}\varepsilon)$ for $\varepsilon$ that is small enough. It follows that the local minimum of the polynomial is 0.

**Exercise 7.49 (!).** Prove the Fundamental Theorem of Algebra: every polynomial $P$ of positive degree has a root in $\mathbb{C}$.

# GEOMETRY 8: Pointwise and uniform convergence

During the work on this sheet, it is allowed to use Tychonoff's Theorem in the following form.

**Theorem.** Let $X$ be a compact topological space, $I$ an arbitrary set, $X^I$ the topological space (in the pointwise convergence topology), of the mappings $I \to X$. Then $X^I$ is compact.

**Exercise 8.1.** Consider the space of of functions from an interval to an interval. Show that the limit of a sequence of continuous functions need not be continuous.

**Definition 8.1.** Let $X$, $Y$ be metric spaces, $\{f_\alpha\}$ a set of continuous functions $X \to Y$. Then $\{f_\alpha\}$ is called *uniformly continuous* if for any $\varepsilon$ there exists $\delta$ such that the image of any $\delta$-ball under any $f_\alpha$ is contained in an $\varepsilon$-ball $B_\alpha$. (Note that $B_\alpha$ can depend upon $\alpha$.)

**Exercise 8.2.** Let $f: X \longrightarrow Y$ be a mapping of metric spaces that maps each Cauchy sequence to a Cauchy sequence. Show that $f$ is continuous as a mapping of topological spaces. Is it true that any continuous mapping maps each Cauchy sequence to a Cauchy sequence?

**Exercise 8.3 (!).** Let $X, Y$ be metric spaces, $\{f_i\}$ a uniformly continuous sequence of continuous functions $X \to Y$. Suppose that $\{f_i\}$ converges to $f$ in the pointwise convergence topology. Show that $f$ is continuous.

**Hint.** Show that $f$ is uniformly continuous, with the same $\varepsilon$, $\delta$ as $\{f_i\}$, and then use the preceding problem.

Pick compact metric spaces $X, Y$, and let $\operatorname{Map}(X, Y)$ be the set of continuous mappings $X \to Y$.

**Exercise 8.4.** For any $f, g \in \operatorname{Map}(X, Y)$ define

$$d_{\sup}(f, g) := \sup_{x \in X} d(f(x), g(x)).$$

Show the correctness of the definition of $d_{\sup}(f, g)$, and that it defines a metric on $\operatorname{Map}(X, Y)$.

**Definition 8.2.** The latter metric is called sup-metric on $\operatorname{Map}(X, Y)$.

**Exercise 8.5 (!).** Let a uniformly continuous sequence of mappings $\{f_i\} \subset \operatorname{Map}(X, Y)$ pointwise converge to $f$. Show that it converges to $f$ in the topology induced by the sup-metric, too.

**Hint.** Let $\sup_{x \in X} d(f(x), f_i(x)) > C$ for any $i$. Find a converging subsequence of $\{x_i\}$'s satisfying $d(f(x_i), f_i(x_i)) > C$. Let $x = \lim_{i \to \infty} x_i$. Due to uniform convergence, $d(f_i(x_i), f_i(x)) \to 0$. Derive a contradiction from the triangle inequality

$$d(f_i(x), f(x)) + d(f_i(x_i), f_i(x)) \geqslant d(f(x), f_i(x_i)).$$

**Exercise 8.6 (!).** (Arzelà-Ascoli Theorem) Let $\Psi \subset \operatorname{Map}(X, Y)$ be closed (w.r.t. the sup-metric) and uniformly continuous. Show that $\Psi$ is compact.

**Hint.** Use Tychonoff's Theorem and the preceding problem. As we have already said, we assume here that $X$ and $Y$ are compact!

**Exercise 8.7 (**).** Find an independent of Tychonoff's Theorem (and thus, of the axiom of choice) proof of Arzelà-Ascoli Theorem.

**Exercise 8.8 (*).** Let $K \subset X$ be compact and $V \subset Y$ open. Denote by $U(K,V) \subset \mathrm{Map}(X,Y)$ the set of all mappings sending $K$ in $V$. Consider the topology on $\mathrm{Map}(X,Y)$, defined by the subbase of all $U(K,V)$. Show that it coincides with the topology induced by the sup-metric.

**Definition 8.3.** The latter topology on $\mathrm{Map}(X,Y)$ is called **topology of uniform convergence**.

**Exercise 8.9.** Show that the pointwise convergence topology is weaker than the uniform convergence topology; in other words that the identity map from $\mathrm{Map}(X,Y)$ endowed with the latter topology onto $\mathrm{Map}(X,Y)$ endowed with the former topology is continuous.

**Definition 8.4.** Let $M$ be a metric space and $Z \subseteq M$. **Diameter** of $Z$ is the number $\mathsf{diam}(Z) := \sup_{x,y\in Z} d(x,y)$.

**Exercise 8.10.** Let $f \in \mathrm{Map}(X,Y)$ be an arbitrary mapping, $\varepsilon$ be a real number, and $\delta(f,\varepsilon)$ be the supremum of $\mathsf{diam}(f(B))$ over all the $\varepsilon$-balls $B$ in $X$. Show that $\lim_{\varepsilon \longrightarrow 0} \delta(f,\varepsilon) = 0$.

**Hint.** Assume that for a convergent to 0 sequence $\varepsilon_i$, a collection of points $x_i \in X$ and a positive constant $C$ one has $\mathsf{diam} f(B_{\varepsilon_i}(x_i))) > C$. Consider a limit point $x$ of $\{x_i\}$. Then each $\varepsilon$-ball around $x$ contains $B_{\varepsilon_i}(x_i)$ (for sufficiently large $i$), implying that the image of this $\varepsilon$-ball has diameter greater than $C$. Thus, $f$ is not continuous.

**Exercise 8.11 (!).** Let $f \in \mathrm{Map}(X,Y)$ be continuous. Show that $f$ is uniformly continuous.

**Hint.** The claim is tautologically equivalent to $\lim_{\varepsilon \longrightarrow 0} \delta(f,\varepsilon) = 0$.

**Exercise 8.12.** Let $\Psi \subset \mathrm{Map}(X,Y)$. Show that $\Psi$ is uniformly continuous if and only if

$$\lim_{\varepsilon \longrightarrow 0} \sup_{f\in\Psi} \delta(f,\varepsilon) = 0.$$

**Exercise 8.13 (*).** Let $d_{\sup}(f,g) < \gamma$. Show that $\delta(f,\varepsilon) < \delta(g,\varepsilon) + \gamma$.

**Exercise 8.14 (*).** Let $\{f_i\}$ be a Cauchy sequence in $(\mathrm{Map}(X,Y), d_{\sup})$. Show that it is uniformly continuous.

**Hint.** We shall show that

$$\lim_{\varepsilon \longrightarrow 0} \sup_i \delta(f_i,\varepsilon) = 0.$$

Using the preceding problem, check that for all $f_i$ in an $\gamma$-ball in $(\mathrm{Map}(X,Y), d_{\sup})$ the numbers $\delta(f_i,\varepsilon)$ differ by no more than $\gamma$. Derive from this that $\sup_i \delta(f_i,\varepsilon) < \delta(f_N,\varepsilon) + \gamma$ for a fixed $N$, and thus

$$\sup_i \delta(f_i,\varepsilon) < \gamma + \max_{i\leqslant N} \delta(f_i,\varepsilon)$$

The limit of the latter, as $\varepsilon \longrightarrow 0$, cannot be greater than $\gamma$, as all $f_i$ are uniformly continuous.

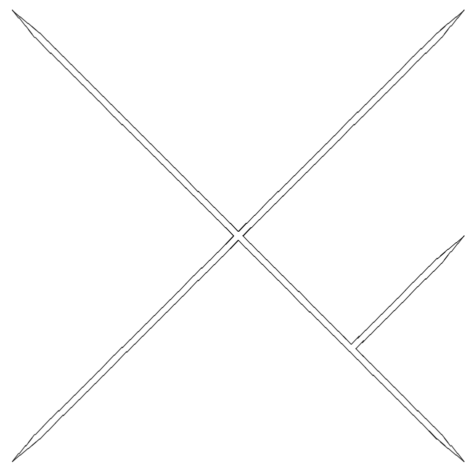
**Exercise 8.15 (*).** Show the completness of the metric space $(\mathrm{Map}(X,Y), d_{\sup})$.

**Exercise 8.16 (*).** Is the space $(\mathrm{Map}(X,Y), d_{\sup})$ locally compact?

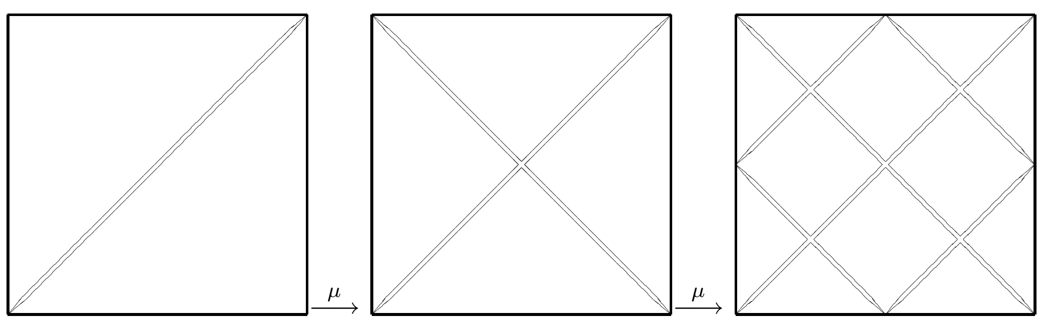
## Peano curve

Let $[a, b] \subset \mathbb{R}$. The mapping $[a, b] \xrightarrow{f} \mathbb{R}^n$ is called **linear**, if $f(\lambda a + (1-\lambda)b) = \lambda f(a) + (1-\lambda)f(b)$, for any $0 < \lambda < 1$. It is called **piecewise linear** if $[a, b]$ is partitioned into subsegments $[a, a_1], [a_1, a_2], [a_2, a_3], \ldots$, and $f$ is linear on each of $[a_\ell, a_{\ell+1}]$. The image of $[a, b]$ under a piecewise linear map is, certainly, a polygonal curve.

Let $f$ be a piecewise linear map $f : [0, 1] \to [0, 1] \times [0, 1]$ satisfying the following property; all the segments of $f([0, 1])$ are parallel either to the line $x = y$ or to the line $x = -y$.



In other words, for any subsegment $[a, a_1]$, on which $f$ linear, $f$ maps $[a, a_1]$ onto a diagonal of a square $Q$, with the sides parallel to the coordinate axes. Let $\mathcal{P}l$ be the space of such piecewise linear mappings. Let us define an operation $\mu$ that produces from an $f \in \mathcal{P}l$ with $k$ linear segments a piecewise linear map with $4k$ linear segments.




Namely $\mu(f)$ is defined as follows.

1. Denote by $a_0, a_1, \ldots, a_k$ the ends of the segments where $f$ was linear. Then $\mu(f)$ maps $a_i$ to $f(a_i)$.

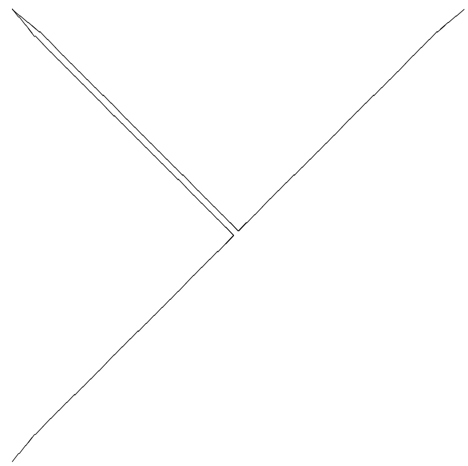
2. Partition each segment $[a_i, a_{i+1}]$ into 4 equal parts:

$$[b_{4i}, b_{4i+1}], [b_{4i+1}, b_{4i+2}], [b_{4i+2}, b_{4i+3}], [b_{4i+3}, b_{4i+4}].$$

   $\mu(f)$ maps $[b_{4i}, b_{4i+1}]$ linearly to $[f(a_i), f\left(\frac{a_i+a_{i+1}}{2}\right)]$, and $[b_{4i+3}, b_{4i+4}]$ to $[f\left(\frac{a_i+a_{i+1}}{2}\right), f(a_{i+1})]$.

3. Consider the square with a diagonal $[f(a_i), f(a_{i+1})]$, and number its vertices clockwise: $f(a_i)$, $A$, $f(a_{i+1})$, $B$. Then $\mu(f)$ maps $[b_{4i+1}, b_{4i+2}]$ linearly to $[f\left(\frac{a_i+a_{i+1}}{2}\right), B]$, and $[b_{4i+2}, b_{4i+3}]$ to $[B, f\left(\frac{a_i+a_{i+1}}{2}\right)]$.

We obtain the following polygonal curve:



**Exercise 8.17.** Consider the segment and the square as metric spaces endowed with the standard metric. Let $f \in \mathcal{P}l$, and the biggest straight segment $[f(a_i), f(a_{i+1})]$ of the corresponding polygonal curve is of length $k$. Then $d_{\sup}(f, \mu(f)) \leqslant \frac{k}{\sqrt{2}}$.

**Exercise 8.18.** Let $f \in \mathcal{P}l$, and the biggest straight segment $[f(a_i), f(a_{i+1})]$ of the corresponding polygonal curve is of length $k$. Then the biggest straight segment in $\mu(f)$ is of length $k/2$.

**Exercise 8.19.** Let $f_0 \in \mathcal{P}l$, $f_1 = \mu(f_0), \dots, f_n = \mu(f_{n-1})$, and the biggest straight segment of the polygonal curve $f([0,1])$ has length $k$. Show that

$$d_{\sup}(f_n, f_{n+1}) < \frac{k}{2^n\sqrt{2}}$$

**Exercise 8.20 (!).** Show that $\{f_i\}$ is a Cauchy sequence in the metric $d_{\sup}$.

**Exercise 8.21.** Let $f \in \mathcal{P}l$, and for all straight segments $[a_i, a_{i+1}]$ $f$ the length of $[f(a_i), f(a_{i+1})]$ is at most

$$\rho(a_{i+1} - a_i),$$

where $\rho > 0$ is a real. Show that $\delta(f, \varepsilon) \leqslant \rho\varepsilon$, where $\delta(f, \varepsilon)$ is the function defined above.

**Exercise 8.22.** Let $f_0 \in \mathcal{P}l$, $f_1 = \mu(f_0), \dots, f_n = \mu(f_{n-1})$, and for all straight segments $[a_i, a_{i+1}]$ $f_0$ the length of $[f(a_i), f(a_{i+1})]$ is at most $\rho(a_{i+1} - a_i)$. Show that $\delta(f_n, \varepsilon) \leqslant \rho 2^n \varepsilon$.

**Exercise 8.23.** Let $f \in \mathcal{P}l$, and the longest straight segment $[f(a_i), f(a_{i+1})]$ of $f([0,1])$ has length $k$. Show that $\delta(\mu(f), \varepsilon) \leqslant 2\frac{k}{\sqrt{2}} + \delta(f, \varepsilon)$.

**Exercise 8.24.** Let $f_0 \in \mathcal{P}l$, $f_1 = \mu(f_0), \dots, f_n = \mu(f_{n-1})$, and the longest straight segment of $f_0([0,1])$ has length $k$. Show that

$$\delta(f_n, \varepsilon) \leqslant 4\frac{k}{2^{n-m}\sqrt{2}} + \rho 2^m \varepsilon \qquad (8.1)$$

for any $n$, $m$ $(n > m)$

**Exercise 8.25.** In the previous problem take $\varepsilon < 2^{-2m}$, $n > 2m$. Derive from (8.1) that

$$\delta(f_n, \varepsilon) \leqslant \frac{4k\sqrt{2} + \rho}{2^{-m}}.$$

Show that for any $i$ the following holds.

$$\delta(f_i, \varepsilon) \leqslant \max\left(\frac{4k\sqrt{2}+\rho}{2^{-m}}, \rho 2^{2m}\varepsilon\right).$$

**Exercise 8.26 (!).** Let $f_0$ linearly map $[0, 1/2]$ to the segment $[(0,0),(1,1)]$, and $[1/2, 1]$ – to $[(1,1),(0,0)]$. Show that $\{f_i\}$ is uniformly continuous.

**Hint.** Derive from the preceding problem that $\lim\limits_{\varepsilon \longrightarrow 0} \sup_i(\delta(f_i, \varepsilon)) = 0$.

**Exercise 8.27.** Derive from Arzelà-Ascoli Theorem the existence of $\lim f_i$ (in sup-metric) and continuity of it as a function $\mathcal{P}: [0,1] \longrightarrow [0,1] \times [0,1]$.

**Definition 8.5.** The function $\mathcal{P}$ defined above is called a **Peano curve**.

**Exercise 8.28.** Find $\mathcal{P}(q)$, for $q = \frac{a}{2^n}$ $(a \in \mathbb{Z})$. (Such numbers are called binary-rational.)

**Exercise 8.29.** Let $Q_2$ be the set of binary-rational numbers. Show that $\mathcal{P}(Q_2)$ is dense on the unit square.

**Exercise 8.30 (!).** Show that $\mathcal{P}([0,1])$ is the whole unit square.

**Hint.** Use the fact that the image of a compact is compact.

**Exercise 8.31 (!).** Is it possible to map, surjectively and continuously, $[0,1]$ onto a cube ? Onto a cube with one point removed?

# GEOMETRY 9: Connectedness

**Definition 9.1.** Let $M$ be a topological space. A closed and open at the same time subset $W \subset M$ is called **clopen**. $M$ without proper clopen subsets is called **connected**. A subset $Z \subset M$ is called **connected** if it is connected in the induced topology.

**Exercise 9.1.** Is $\mathbb{R}$ connected?

**Exercise 9.2 (!).** Let $X$, $Y$ be connected. Show that $X \times Y$ is connected.

**Hint.** Let $U \subseteq X \times Y$ be clopen. Consider $U \cap X \times \{y\}$. Show that $X \times \{y\}$ (in induced topology) is homeomorphic to $X$, and $U \cap X \times \{y\}$ is clopen there.

**Exercise 9.3.** Is $\mathbb{R}^n$ connected (in its natural topology)?

**Exercise 9.4.** Assume that it is possible to connect any two points $x, y$ in $M$ by a path, that is, to find a continuous mapping $[0,1] \xrightarrow{\varphi} M$ satisfying $\varphi(0) = x$, $\varphi(1) = y$. Show that $M$ is connected.

**Remark.** Such an $M$ is called **path-connected**.

**Exercise 9.5.** Remove a point from a circle or the plane. Show that the result is connected.

**Exercise 9.6 (!).**

a. Remove a finite number of points from $\mathbb{R}^2$. Show that the result is connected.

b. Remove a point from an interval. Show that the result is not connected.

**Exercise 9.7 (!).** Show that the following spaces are not homeomorphic to each other: $\mathbb{R}$, $\mathbb{R}^2$, the circle.

**Exercise 9.8 (!).** Show that the following spaces are not homeomorphic to each other: closed interval, half-open interval, open interval.

**Exercise 9.9.** Let $f: X \longrightarrow Y$ be continuous and $X$ be connected. Show that $f(X)$ is connected.

**Exercise 9.10 (!).** Let $U \subseteq [0,1]$ be connected. Show that $U$ is either a closed interval, or a half-open interval, or an open interval.

**Exercise 9.11.** Let $f: X \longrightarrow \mathbb{R}$ be continuous and $X$ be connected. Assume that $f$ takes positive as well as negative values. Show that $f(x) = 0$ for some $x \in X$.

**Exercise 9.12 (*).** Let $M$ be a connected metrizable countable topological space. Show that $M$ consists of one point.

**Exercise 9.13.** Show that the union of two connected subsets of a topological space $M$ is connected, provided that their intersection is nonempty.

**Exercise 9.14 (!).** Let $x \in M$ and $W$ be the union of all the connected subsets of $M$ containing $x$. Show that $W$ is connected.

**Definition 9.2.** In such a situation $W$ is called **the connected component** of $x$ (or just **a connected component**).

**Exercise 9.15.** Show that $W \subset M$ is a connected component if and only if any connected subset containing $W$ coincides with $W$.

**Exercise 9.16.** Show that $M$ is the disjoint union of its connected components.

**Exercise 9.17.** Show that each connected component of $M$ is closed.

## Totally disconnected spaces

**Definition 9.3.** A topological space $M$ is called **totally disconnected** if each connected component of $M$ consists of one point.

**Exercise 9.18.** Show that $\mathbb{Q}$, the space of rational numbers, in the topology induced by $\mathbb{R}$, is totally disconnected, but not discrete.

**Exercise 9.19 (*).** Show that $\mathbb{Q}_p$, the space of $p$-adic numbers, is totally disconnected.

**Exercise 9.20 (*).** Show that the product of totally disconnected spaces is totally disconnected.

**Exercise 9.21.** Let $S$ be a subbase in a Hausdorff topological space $M$, and all the elements of $S$ clopen. Show that $M$ is totally disconnected.

**Exercise 9.22 (!).** Consider the set $\{0,1\}$ equipped with the discrete topology. Let $\{0,1\}^I$ be the product of $I$ copies of $\{0,1\}$ with Tychonoff topology, with $I$ being an arbitrary index set. Show that $\{0,1\}^I$ is totally disconnected.

**Hint.** Use the preceding problem.

**Exercise 9.23 (*).** Let $M$ be Hausdorff topological space, $M_1$ be the sets of connected components of $M$, and $M \xrightarrow{\pi} M_1$ the natural projection (each point is mapped to its connected component). On $M_1$ introduce the following topology: $U \subset M_1$ open if $\pi^{-1}(U) \subset M$ is open. Show that $M_1$ is totally disconnected. Show that any continuous mapping $M \xrightarrow{\pi_2} M_2$ from $M$ to a totally disconnected space $M_2$ can be written as a composition of continuous mappings $M \xrightarrow{\pi} M_1 \longrightarrow M_2$.

**Hint.** If $S \subset M_1$ is connected then the preimage $\pi^{-1}(S)$ is connected, too. Indeed, if $W \subset \pi^{-1}(S)$ is clopen then $W = \pi^{-1}(W_1)$ (if $W$ intersects a connected component of $M$, then $W$ contains it). Thus $W_1$ is clopen.

**Exercise 9.24.** Let $U$ be an open subset of a compact Hausdorff space and a collection of closed subsets $\{K_i\}$, so that their intersection is conatined in $U$. Show that $\{K_i\}$ contains a finite subcollection so that their intersection is contained in $U$.

**Exercise 9.25 (*).** Let $M$ be a totally disconnected compact Hausdorff space. Show that, for each point $x \in M$, the intersection of all the clopen subsets of $M$ containing $x$ is $\{x\}$.

**Hint.** Let $P$ be the intersection of the clopen subsets containing $x$. Obviously $P$ is closed. Show that $P$ is either $\{x\}$ or disconnected. In the latter case $P$ is the disjoint union of two nonempty closed subsets $P_1$, $P_2$. As T4 holds in $M$ (show this), find for $P_1$, $P_2$ nonintersecting open neighbourhoods $U_1$, $U_2$. Derive from the preceding problem that $U_1 \cup U_2$ contains a clopen subset $W \subset M$ containing $x$. Show that $W \cap U_i$ are clopen, and derive from this that $P = \{x\}$.

**Exercise 9.26 (*).** Let $M$ be a totally disconnected compact Hausdorff space. Show that the clopen subsets form a base of the topology of $M$.

**Hint.** Let $U \subset M$ be open and $x \in U$. For each point in $M \backslash U$ pick a clopen neighbourhood that does not contain $x$ (show that this is always possible). This is a cover $\{U_\alpha\}$ of $M \backslash U$. As $M \backslash U$ is compact, $\{U_\alpha\}$ contains a finite subcover $U_1, ...U_n$. Show that the complement to $\cup U_i$ is clopen, contains $x$, and is contained in $U$.

**Exercise 9.27 (*).** Let $M$ be a totally disconnected compact Hausdorff space. Let $x, y \in M$ be two distinct points. Show that $M$ admits a continuous mapping to $\{0, 1\}$ (with discrete topology) such that $x$ goes to 0 and $y$ goes to 1.

**Exercise 9.28 (*).** Let $M$ be a totally disconnected compact Hausdorff space. Let $I$ be the set of all continuous mappings from $M$ to $\{0, 1\}$. Define a natural mapping $M \longrightarrow \{0, 1\}^I$. Show that it is a continuous embedding, and that the image of $M$ is closed.

**Exercise 9.29 (*).** Let $M$ be a compact Hausdorff space. Show that the following statements are equivalent.

(i) $M$ is totally disconnected.

(ii) $M$ can be embedded into $\{0, 1\}^I$ for some set $I$ of indices.

**Remark.** Recall that if a compact $M$ admits a continuous injective mapping $f : M \to X$ into a Hausdorff space $X$ then $f$ is a homeomorphism between $M$ and $f(M) \subset X$ with induced topology.

# Geometry 10: The fundamental group and the loop space

## Path-connectedness

**Definition 10.1.** Let $M$ be a topological space. Recall that a **path** in $M$ is a continuous mapping $[a,b] \xrightarrow{\varphi} M$. In this case one says that the path $\varphi$ **connects the points** $\varphi(a)$ **and** $\varphi(b)$. $M$ is called **path-connected** when any two points of $M$ can be connected by a path $[a,b] \xrightarrow{\varphi} M$.

**Exercise 10.1.** Let $a$, $b$, $c$ are points in $M$, so that $a$ can be connected (by a path) to $b$, and $b$ can be connected to $c$. Show that $a$ can be connected to $c$.

**Exercise 10.2.** From this derive that a union of path-connected subsets of $M$ containing a point $x \in M$ is path-connected.

**Definition 10.2.** The union of all the subsets of $M$, containing a fixed point $x$ is called a **path-connected component** of $M$

**Exercise 10.3.** Consider $X \subset \mathbb{R}^2$ that is the union of the graph of the function $\sin(1/t)$ and the segment $[(0,1),(0,-1)]$. Show that $X$ is locally compact, connected, but not path-connected. Find its path-connected components.

**Exercise 10.4 (*).** Construct a compact, connected metrizable topological space with infinitely many path-connected components.

**Definition 10.3.** Let $\{M_\alpha\}$ be a collection of topological spaces indexed by the set $\mathfrak{A}$. The disjoint union $\coprod_{\alpha\in\mathfrak{A}} M_\alpha$ is a topological space whose points are pairs $(\alpha, m) \mid \alpha \in \mathfrak{A}, m \in M_\alpha$, and a base of the topology is given by the open sets in all $M_\alpha$.

**Exercise 10.5.** Show that the disjoint union of one-point spaces is discrete. Show that the natural projection $\coprod_{\alpha\in\mathfrak{A}} M_\alpha \longrightarrow \mathfrak{A}$ on $\mathfrak{A}$ with discrete topology is continuous.

**Definition 10.4.** A topological space $M$ is called locally connected (respectively, locally path-connected), if any point $x \in M$ is contained in a connected (respectively, path-connected) open set.

**Exercise 10.6.** Let $M$ be a topological space. Show that $M$ is locally connected (resp. locally path-connected) iff $M$ is a disjoint union of its (path-)connected components.

**Exercise 10.7.** Show that a connected space is path-connected iff it is locally path-connected.

**Exercise 10.8.** Show that an open subset of $\mathbb{R}^n$ is locally path-connected.

**Exercise 10.9 (**).** Let $\omega$ be the smallest uncountable ordinal, and $\varphi: [0,1] \longrightarrow \omega$ the corresponding bijection. Let $X \subset [0,1]\times[0,1]$ be the subset of the square constisting of $x, y$ satisfying $\varphi(x) > \varphi(y)$. Show that $X$ is connected. Show that path-connected components of $X$ are either points or segments of horisontal intervals.

**Hint.** Show that the intersection of $X$ with any vertical segment is nowhere dense. Let $V \subset [0,1]\times[0,1]$ be a connected closed subset of the square contained in $X$. Show that $V$ intersects each vertial segment in no more than 1 point. Thus $V$ is the graph of a continuous mapping $\gamma: [a,b] \longrightarrow [0,1]$, satisfying $\varphi(\gamma(a)) < \varphi(a)$. Show that this mapping is constant.

**Definition 10.5.** Let $M$ be a complete locally compact metric space. Recall that a **geodesic** in $M$ is a metric-preserving mapping $[a, b] \longrightarrow M$. We say that $M$ is **geodesically connected** if any two points can be connected by a geodesic. Obviously such a space is path-connected.

**Definition 10.6.** Let $M$ be a complete locally compact metric space. We say that $M$ is **Lipschitz connected**, with Lipschitz constant $C \geqslant 1$, if for any $x, y \in M$ and any $\varepsilon > 0$ there exists a sequence of points $x = x_1, x_2, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) < \varepsilon$, $\sum_i d(x_i, x_{i+1}) \leqslant Cd(x, y)$. In other words, one can place $n$ points between $x$ and $y$ so that they are at distance at most $\varepsilon$ from each other, whereas the length of the polygonal line they form is at most $Cd(x, y)$.

**Exercise 10.10 (*).** Show that any geodesically connected metric space is Lipschitz connected with Lipschitz constant 1.

**Hint.** This is Hopf-Rinow Theorem.

**Exercise 10.11 (!).** Let $(M, d)$ be a Lipshitz connected metric space, with constant $C$. Define a function $d_h: M \times M \longrightarrow \mathbb{R}$ as

$$\lim_{\varepsilon \longrightarrow 0} \inf \left(\sum d(x_i, x_{i+1})\right),$$

where inf is taken over such sequences $x = x_1, x_2, \ldots, x_n = y$ that $d(x_i, x_{i+1}) < \varepsilon$. Show that $d(x, y) \leqslant d_h(x, y) \leqslant Cd(x, y)$ for any $x, y \in M$. Show that $d_h$ is a metric and that $(M, d)$ is homeomorphic to $(M, d_h)$.

**Exercise 10.12 (*).** Show that $(M, d_h)$ is Lipschitz connected, for any $C > 1$.

**Exercise 10.13 (*).** Show that $(M, d_h)$ satisfies Hopf-Rinow condition (and is therefore geodesically connected).

**Definition 10.7.** Recall that a mapping $[a, b] \xrightarrow{\varphi} M$ **satisfies the Lipschitz condition, with constant** $C > 0$, if $d(\varphi(x), \varphi(y)) \leqslant C|x - y|$ for any $x, y \in [a, b]$. It is easy to see that a Lipshitz mapping is continuous.

**Exercise 10.14 (*).** Let $M$ be a complete locally compact metric space. Show that $M$ is Lipschitz connected with constant $C$ iff one can connect any two points by a Lipschitz path with the same (universal for $M$) constant.

**Hint.** Use the previous problem and the inequality $d(x, y) \leqslant d_h(x, y) \leqslant Cd(x, y)$.

**Remark.** We have established that a Lipschitz connected metric space is path-connected.

**Exercise 10.15.** Consider the circle $S$ on the plane with induced metric. Show that $S$ is Lipschitz connected with constant $\frac{\pi}{2}$.

**Exercise 10.16 (*).** Show that $\frac{\pi}{2}$ is the smallest possible constant for which the circle with such a metric is Lipschitz connected.

**Exercise 10.17 (**).** Consider the mapping $]0, \infty[ \longrightarrow \mathbb{R}^2$, given in polar coordinates by the function $\theta = 1/x$, $r = x$ (this is a spiral winding around 0 with the step $\frac{1}{2\pi n}$). Let $X$ be the closure of the graph of this function (that obviously consists of the graph itself and 0). Show that $X$ is path-connected. Show that $X$ is not Lipschitz connected, no matter what constant $C$ we take.

**Exercise 10.18 (*).** Let $M$ be a locally compact complete metric space. Denote by $S_\varepsilon(x)$ the sphere of radius $\varepsilon$ with the centre in $x$. Show that the following conditions are equivalent.

(i) $M$ is Lipschitz connected, with constant $C$

(ii) for any $x, y \in M$ and any $r_1, r_2 > 0$ satisfying $r_1 + r_2 \leqslant 1$, the distance between $S_{dr_1}(x)$ and $S_{dr_2}(y)$ is not bigger than $Cd(1 - r_1 - r_2)$, where $d = d(x, y)$.

**Hint.** To derive (ii) from Lipschitz connectedness, consider a Lipschitz curve through $x, y$. Lipschitz connectedness follows immediately from (ii). The distance from $x$ to $S_{d(1-C^{-1}\varepsilon)}(y)$ is at most $\varepsilon$; take as $x_2$ the point of the sphere realizing this distance (that is possible, as the sphere is compact by Hopf-Rinow Theorem), and use induction.

**Remark.** Recall that in one version that Hopf-Rinow condition says that the distance between $S_{dr_1}(x)$ and $S_{dr_2}(y)$ equals $d(1 - r_1 - r_2)$.

## Loop space

**Definition 10.8.** Let $(M, x)$ be a topological space with a marked point $x$. Consider the set $\Omega(M, x)$ of paths $[0, 1] \xrightarrow{\varphi} M$, $\varphi(0) = \varphi(1) = x$, with open-compact topology (the base of this topology consists of the set $U(K, W)$ of mappings of a given compact $K \subset [0, 1]$ into a given open set $W \subset M$). The $\Omega(M, x)$ is called **the loop space** for $(M, x)$.

**Exercise 10.19 (!).** Let $M$ be metrizable. Show that $\Omega(M, x)$ is metrizable too, with the metric

$$d(\gamma, \gamma') = \sup_{x \in [0,1]} d(\gamma(x), \gamma'(x)).$$

**Exercise 10.20.** Let $(M, x)$ be a space with a marked point $x$. $M_0$ the connected component of $x$, and $M_1$ the path-connected component of $x$. Show that $\Omega(M, x) = \Omega(M_0, x) = \Omega(M_1, x)$.

**Exercise 10.21.** Let $X, Y$ be compacts, $\mathcal{W}$ be the space of mappings from $X$ to $M$ endowed with open-compact topology. Construct a bijection between continuous mappings from $Y$ to $\mathcal{W}$ and continuous mappings $X \times Y \longrightarrow M$.

**Exercise 10.22 (!).** Let $\gamma, \gamma' \in \Omega(M, x)$. Construct a bijection between the following sets:

(i) Paths $\Gamma : [0, 1] \longrightarrow \Omega(M, x)$, connecting $\gamma$ and $\gamma'$.

(ii) Continuous mappings $\Psi$ from the square $[0, 1] \times [0, 1]$ to $M$ that map $\{1\} \times [0, 1]$ to $x$ and such that $\Psi\Big|_{[0,1]\times\{0\}} = \gamma$, $\Psi\Big|_{[0,1]\times\{1\}} = \gamma'$.

**Definition 10.9.** Paths $\gamma, \gamma' \in \Omega(M, x)$ for which the mappings $\Psi : [0, 1] \times [0, 1] \longrightarrow M$ exist, are called **homotopic**, and $\Psi$, that connectes them, is called **homotopy**.

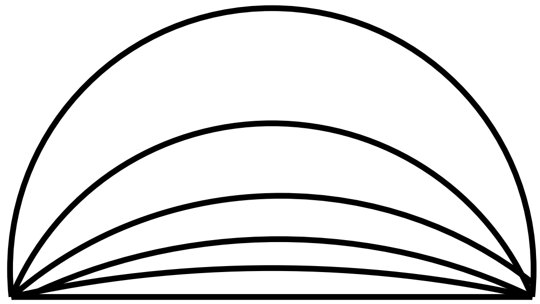
**Exercise 10.23.** Show that the set of loops homotopic to $\gamma \in \Omega(M, x)$ is a path-connected component of $\gamma \in \Omega(M, x)$.

**Exercise 10.24.** Show that the homotopy of loops is an equivalence relation.

**Remark.** Loops homotopic to each other are also called **homotopy equivalent**.

**Definition 10.10.** Let $(M, x)$ be path-connected. The set of homotopy equivalent classes of loops is denoted by $\pi_1(M, x)$.

**Exercise 10.25 (*).** Let $M \subset \mathbb{R}^2$ be the union of the closed segment $[(0,1),(0,-1)]$ and arcs of circles of diameters $3, 4, 5, \ldots$ that connect $(0,1)$ and $(0,-1)$.



Show that $M$ is path-connected. Show that for any $x \in M$ the space $\Omega(M, x)$ is not locally path-connected.

**Exercise 10.26 (*).** Let $(M, d)$ be a geodesically connected locally compact metric space such that for a $\delta > 0$ and any $x, y \in M$, $d(x, y) < \delta$, the geodesic connecting $x$ and $y$ is unique. Let $\Delta_\delta \subset M \times M$ be the set of pairs $x, y \in M$, $d(x, y) < \delta$. Consider the mapping $\Delta_\delta \longrightarrow M$ of pairs to the middle points of the geodesics that connect pairs. Show that it is continuous.

**Hint.** Let $\{(x_i, y_i)\}$ is a sequence of pairs converging to $(x, y)$, and $\{z_i\}$ the sequence of middle points of corresponding geodesics. Due to local compactness, $\{z_i\}$ has limit points and does not contain infinite discrete subsets. Any limit point of $\{z_i\}$ will be the middle of geodesic connecting $x$ and $y$. Thus $\{z_i\}$ has unique limit point.

**Exercise 10.27 (*).** Consider the mapping $\Delta_\delta \otimes [0,1] \xrightarrow{\Psi} M$, of pairs $x, y \in M$, $d(x, y) = d$, $t \in [0,1]$ to points $\gamma_{x,y}\left(\frac{t}{d}\right)$, where $\gamma_{x,y}$ is a geodesic connecting $x$ and $y$ (when $x = y$ set $\Psi(x, y, t) = x$). Show that this mapping is continuous.

**Hint.** Use the previous problem and the construction of a geodesic as the limit of middle points of segments used in the proof of Hopf-Rinow Theorem.

**Definition 10.11.** Let $M$ be a metric space. A path $\gamma: [0,1] \longrightarrow M$ is called **piecewise geodesic** if $[0,1]$ is subdivided into $[0, a_1]$, $[a_1, a_2]$, ..., $[a_n, 1]$, and on each of these closed intervals $\gamma$ satisfies $d(\gamma(x), \gamma(y)) = \lambda_i |x - y|$, for some constant $\lambda_i$

**Remark.** If $M$ is an open set in $\mathbb{R}^n$ with the natural metric then, as shown is Sheet 4, geodesics are segments. Thus piecewise geodesics are piecewise linear. Such mappings are also called **piecewise linear**.

**Exercise 10.28 (*).** In the conditions of Excercise 10.26, consider $\Omega(M, x)$ as a metric space (with sup-metric). Show that any loop $\gamma \in \Omega(M, x)$ is homotopic to a piecewise geodesic, so that the homotopy can be chosen in any $\varepsilon$-neighbourhood $B_\varepsilon(\gamma) \subset \Omega(M, x)$.

**Exercise 10.29 (*).** Derive from this that $\Omega(M, x)$ is locally path-connected.

**Remark.** In such a situation $\pi_1(M, x)$ is the set of connected components of $\Omega(M, x)$.

**Exercise 10.30.** Let $M$ be an open set in $\mathbb{R}^n$. Show that $\Omega(M, x)$ is locally path-connected.

**Hint.** Show that any loop can be homotopically deformed, in a sufficiently small $\varepsilon$-neighbourhood, into a piecewise linear.

## Fundamental group

**Exercise 10.31.** Given loops $\gamma_1, \gamma_2 \in \Omega(M, x)$, consider the loop $\gamma_1\gamma_2 \in \Omega(M, x)$, defined as follows:

$$\gamma_1\gamma_2(\lambda) = \begin{cases} \gamma_1(2\lambda) & \lambda \in [0, 1/2], \\ \gamma_2(2\lambda - 1) & \lambda \in [1/2, 1]. \end{cases}$$

Show that the class of the homotopy $\gamma_1\gamma_2$ depends only on classes of homotopies $\gamma_1$, $\gamma_2$: if $\gamma_1 \sim \gamma_1'$, $\gamma_2 \sim \gamma_2'$ then $\gamma_1\gamma_1' \sim \gamma_2\gamma_2'$.

**Exercise 10.32.** Show that $(\gamma_1\gamma_2)\gamma_3$ is homotopy equivalent to $\gamma_1(\gamma_2\gamma_3)$.

**Exercise 10.33.** Given a loop $\gamma \in \Omega(M, x)$, denote by $\gamma^{-1}$ the loop $\gamma^{-1}(x) = \gamma(1-x)$. Show that the loops $\gamma\gamma^{-1}$ and $\gamma^{-1}\gamma$ are homotopic to the trivial loop $[0, 1] \longrightarrow x$.

**Remark.** Loops that are homotopic to the trivial one are called **null-homotopic**.

**Exercise 10.34 (!).** Show that the operation $\gamma_1, \gamma_2 \longrightarrow \gamma_1\gamma_2$ makes $\pi_1(M, x)$ into a group.

**Definition 10.12.** This group is called **the fundamental group** of $M$.

**Exercise 10.35.** Let $X \xrightarrow{f} Y$ be a continuous mapping of path-connected spaces, and $x \in X$. Consider the corresponding mapping

$$\Omega(X, x) \xrightarrow{\check{f}} \Omega(Y, f(y)), \qquad \gamma \mapsto \gamma \circ f.$$

Show that $\check{f}$ maps homotopic paths to homotopic and induces a homomorphism of fundamental groups.

**Exercise 10.36.** Let $M$ be a path-connected topological space, and $x, y \in M$. Consider the space $\Omega(M, x, y)$ of paths $[0, 1] \longrightarrow M$ connecting $x$ and $y$ with open-compact topology. As above, paths are called homotopic (homotopy equivalent) is they lie in the same path-connected component of $\Omega(M, x, y)$. Define an operation $\Omega(M, x, y) \times \Omega(M, y, z) \longrightarrow \Omega(M, x, z)$, $\gamma_1, \gamma_2 \mapsto \gamma_1\gamma_2$ using the same formula as in Excercise 10.31. Show that this mapping is continuous and maps homotopic paths to homotopic.

**Exercise 10.37 (!).** Let $x, y \in M$, and $\gamma_{xy}[0, 1] \longrightarrow M$ be a path connecting $x$ and $y$. Define $\gamma_{xy}^{-1}$ using $\gamma_{xy}^{-1}(\lambda) = \gamma_{xy}(1 - \lambda)$. Consider the mapping $\Omega(M, x) \longrightarrow \Omega(M, y)$, $\gamma \mapsto \gamma_{xy}^{-1}\gamma\gamma_{xy}$ and $\Omega(M, y) \longrightarrow \Omega(M, x)$, $\gamma \mapsto \gamma_{xy}\gamma\gamma_{xy}^{-1}$. Show that these mappings map homotopic paths to homotopic. Let $f$, $g$ be corresponding maps on fundamental groups. Show that $f$ $g$ are inverses of each other and induce an isomorphism of groups $\pi_1(M, x) \xrightarrow{\varphi_{\gamma_{xy}}} \pi_1(M, y)$.

**Remark.** As can be seen from the preceding problem, if $\pi_1(M)$ is not abelian then the isomorphism $\pi_1(M, x) \cong \pi_1(M, y)$ obtained there nontrivially depends upon the choice of the path $\gamma_{xy}$. Nevertheless, when the dependence upon the marked point is not important, the fundamental group $M$ is denoted simply by $\pi_1(M)$. This notation is not quite correct.

**Exercise 10.38 (!).** In conditions of the preceding problem, let $x = y$, and $\gamma_{xx}$ a path. Show that the isomorphism $\pi_1(M,x) \xrightarrow{\varphi_{\gamma_{xx}}} \pi_1(M,x)$ obtained above can be expressed via $\gamma_{xx}$ as follows: $\gamma \longrightarrow \gamma_{xx}\gamma\gamma_{xx}^{-1}$.

## Simply connected spaces

**Definition 10.13.** Let $M$ be a path-connected topological space. We say that $M$ is **simply connected** when all the loops on $M$ are contractible, i.e. when $\pi_1(M) = \{1\}$.

**Exercise 10.39.** Show that $\mathbb{R}^n$ is simply connected.

**Definition 10.14.** Let $(M,x)$ be a topological space with a marked point, $M \times [0,1] \xrightarrow{\varphi} M$ be a continuous mapping such that $\varphi(M \times \{1\}) = \{x\}$ and $\varphi\Big|_{M\times\{0\}}$ the identity mapping from $M = M \times \{0\}$ to $M$. Then $(M,x)$ is called **contractible**. In such a situation one says that $\varphi$ **defines a homotopy between the identity mapping and the projection** $M \longrightarrow \{x\}$**.**

**Exercise 10.40 (!).** Let $(M,x)$ be path-connected and contractible. Show that for any point $y \in M$ the space $(M,y)$ is contractible.

**Hint.** Let $M \times [0,1] \xrightarrow{\varphi} M$ be a homotopy between the identity mapping and the projection onto $\{x\}$, and $[1,0] \xrightarrow{\gamma} M$ be a path connecting $x$ and $y$. Take $M \times [0,1] \xrightarrow{\varphi_1} M$, mapping $(m,t)$ to $\varphi(m,2t)$ for $t \in [0,1/2]$ and $(m,t)$ in $\gamma(2t-1)$ for $t \in [1/2,0]$.

**Exercise 10.41.** Show that a contactible topological space is path-connected.

**Remark.** Two preceding problems immediately imply that the contractibility of $(M,x)$ does not depend upon the choice of $x$. Thus we say in the remainder simply "$M$ is contractible".

**Exercise 10.42.** Show that a contractible space is simply connected.

**Exercise 10.43 (!).** Let $V \subset \mathbb{R}^n$ be a **star subset** of $(\mathbb{R}^n, x)$, that is, it satisfies the property that any line through $x \in \mathbb{R}^n$ intersects $V$ in a connected set, and $x \in V$. Show that $V$ is contractible.

**Exercise 10.44.** Let $V \subset \mathbb{R}^n$ be a convex set. Show that it is contractible.

**Definition 10.15.** Let $N$ be a subset of a toplogical space $M$. The **deformation retract** (or simply **retract**) of $M$ to $N$ is a continuous mapping $M \times [0,1] \xrightarrow{\varphi} M$, such that $\varphi(M \times \{1\}) \subset N$, its restriction onto $N$ the identity, and $\varphi\Big|_{M\times\{0\}}$ an identity mapping. In this case $N$ is called a **retract** of $M$.

**Exercise 10.45 (!).** Let $N$ be a retract of $M$, and $n \in N$. Show that the natural mapping $\pi_1(N,n) \longrightarrow \pi_1(M,n)$ is an isomorphism.

**Definition 10.16.** Let $M$ be a topological space, and $\sim$ be an equivalence relation. As always, the set of equivalence classes is denoted by $M/\sim$. We introduce on $M/\sim$ the **quotient topology**: the open subsets of $M/\sim$ are those, whose preimages in $M$ are open. In particular, if $G$ is a group acting on $M$, there is the natural (orbit) equivalence relation on $M$: $x \sim y$ if there exists $g \in G$ satisfying $g \cdot x = y$. The quotient of $M$ w.r.t. this equivalence relation is called the quotient space w.r.t. the $G$-action and denoted by $M/G$. The corresponding equivalence classes are called $G$-**orbits** of $M$.

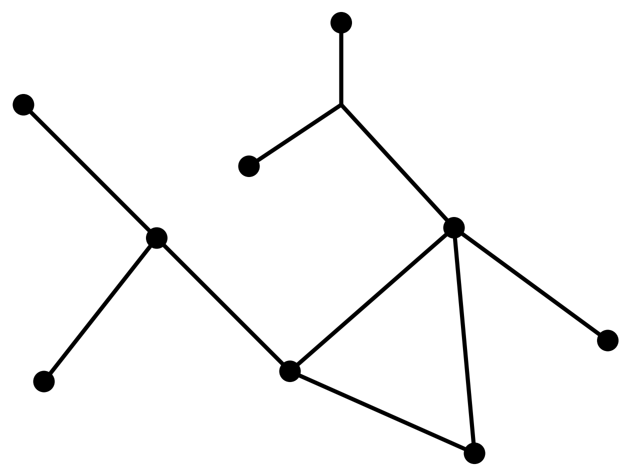
**Exercise 10.46.** Let $M$ be a Hausdorff topological space and $\{x_1, \dots, x_n\} \subset M$ and $\{y_1, \dots, y_m\} \subset M$ two disjoint finite subsets. Show that for these subsets there exist non-intersecting neighbourhoods.

**Exercise 10.47 (!).** Let $M$ be a Hausdorff topological space and $G$ a finite group of $M$-homeomorphisms. Show that $M/G$ is Hausdorff.

**Hint.** Let $x, y$ be two points in distinct $G$-orbits. Find non-intersecting $G$-invariant neighbourhoods of $x$ and $y$. For this, apply Excercise 10.46 to the orbits $Gx$, $Gy$, obtain neighbourhoods $U$, $U'$, and pick $\bigcap_{g \in G} gU$, $\bigcap_{g \in G} gU'$.

**Exercise 10.48 (*).** Give an example of a Hausdorff space $M$ and non-Hausdorff space $M/G$ (here the group $G$ will be infinite).

**Definition 10.17.** Let $\Gamma$ be a graph, that is, a data collection consistiting of "vertex set" $\{\mathcal{V}\}$ and "edge set" $\{\mathcal{R}\}$, and information on which vertices are endpoints of which edges.



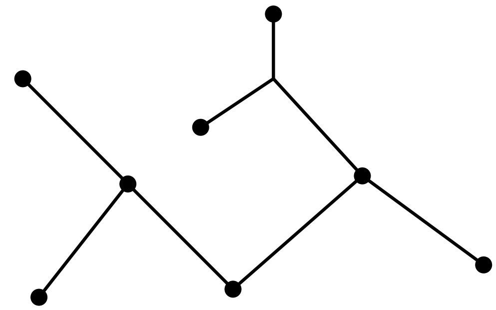
More precisely, one may define $\Gamma$ as a pair of sets $\mathcal{V}$, $\mathcal{R}$ and a surjection $\{\mathcal{R}\} \times \{/, \infty\} \stackrel{-}{\longrightarrow} \{\mathcal{V}\}$. Introduce on $\{\mathcal{R}\} \times [/, \infty]$ the equivalence relation generated by the following: endpoints of two edges are equivalent if the are incident to the same vertex. This relatin glues together endpoints of edges through the same vertex. The quotient $\{\mathcal{R}\} \times [/, \infty]$ w.r.t. this equivalence relation is called the **topological space of the graph**.

**Exercise 10.49.** Show that the topological space of any graph is Hausdorff.

**Exercise 10.50.** A graph is called connected if any vertex is connected to any other vertex by a sequence of edges. Show that the topological space of a connected graph is path-connected.

**Exercise 10.51 (**).** Let $\Gamma$ be a graph with infinite vertex set. Show that $\Gamma$ contains either an infinite clique (i.e. the set of pairwise connected by edges vertices), or an infinite coclique (i.e. the set of vertices such that none of them are connected by an edge).

**Exercise 10.52 (!).** Let $\Gamma$ be a connected graph with $n$ vertices and $n-1$ edges (such a graph is called a **tree**).



Show that the topological space $M_\Gamma$ of $\Gamma$ is contractible.

**Exercise 10.53 (*).** Let $\Gamma$ be an infinite graph so that each of its connected finite subgraphs is a tree. Show that $\pi_1(M_\Gamma) = \{1\}$.

**Exercise 10.54 (*).** Let $S^n$ be an $n$-dimensional sphere ($n > 1$). Show that $S^n$ is simply connected.

**Hint.** Use geodesic connectedness.

## Coverings

**Definition 10.18.** Let $\tilde{M} \xrightarrow{\pi} M$ be a continuous mapping of topological spaces; $\pi$ is called a **covering** when any point has a neighbourhood $U$ such that $\pi^{-1}(U)$ is the product of $U$ and a discrete topological space $K$, so that the natural mapping $\pi^{-1}(U) \xrightarrow{\pi} U$ coincided the projection $\pi^{-1}(U) = U \times K \longrightarrow U$. In this case one also says that $\tilde{M}$ **covers** $M$.

We consider the circle $S^1$ as the quotient $S^1 = \mathbb{R}/\mathbb{Z}$. This gives a natural group structure on $S^1$.

**Exercise 10.55.** Let $n \neq 0$ be an integer. Condider a natural mapping $S^1 \longrightarrow S^1$, $t \longrightarrow nt$. Show that it is a covering.

**Exercise 10.56.** Show that the natural projection $\mathbb{R} \longrightarrow S^1 = \mathbb{R}/\mathbb{Z}$ is a covering.

**Exercise 10.57.** Show that the natural projection $\mathbb{R}^n \longrightarrow (S^1)^n$ is a covering.

**Exercise 10.58.** Consider the quotient $S^n \longrightarrow S^n/\{\pm 1\} = \mathbb{R}P^n$ of the sphere w.r.t. the central symmetry, with the natural topology. Show that it is a covering.

**Exercise 10.59.** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering, and $\tilde{M}' \subset M$ a subspace that covers $M$, too. Show that $\tilde{M}'$ is clopen in $\tilde{M}$.

**Exercise 10.60.** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering, and $M$ path-connected. Show that $\tilde{M}$ is locally path-connected. Show that any path-connected component of $\tilde{M}$ covers $M$.

**Exercise 10.61 (!).** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering, and $M$ path-connected. Show that $\tilde{M}$ is connected iff it is path-connected.

**Definition 10.19.** Let $\gamma : [a, b] \longrightarrow M$ be a path, and $\tilde{M} \xrightarrow{\pi} M$ a covering of $M$. A mapping $\tilde{\gamma} : [a, b] \longrightarrow \tilde{M}$ is called a **lifting of** $\gamma$ if $\tilde{\gamma} \circ \pi = \gamma$.

**Exercise 10.62 (!).** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering, and $\gamma : [a,b] \longrightarrow M$ a path joining $x$ and $y$. Show that for any $\tilde{x} \in \pi^{-1}(\{x\})$ the lifting $\tilde{\gamma}$, mapping $a$ to $\tilde{x}$, exists, and is unique.

**Exercise 10.63 (!).** Show that homotopic paths are lifted to homotopic paths, and that $\tilde{\gamma}(y) \in \pi^{-1}(\{y\})$ is uniquely determined by the class of the homotopy $\gamma$ in $\Omega(M, x, y)$ and the point $\tilde{x}$.

**Remark.** Denote by $\pi_1(M, x, y)$ the set of classes of homotopic paths from $x$ to $y$. We have a mapping

$$\pi^{-1}(\{x\}) \times \pi_1(M, x, y) \xrightarrow{\Psi} \pi^{-1}(\{y\})$$

**Definition 10.20.** Let $\tilde{M} \xrightarrow{\pi} M$ be a cover, and $M$ path-connected. The space $\tilde{M}$ is called a **universal cover** if it is connected and simply connected.

**Remark.** Simple connectedness was defined for path-connected spaces only. But this does not present an obstacle, as it follows from the Excercise 10.61 that $\tilde{M}$ is path-connected.

**Exercise 10.64 (!).** Let $\tilde{M} \xrightarrow{\pi} M$ be a universal cover. Fix $x \in M$ and $\tilde{x} \in \pi^{-1}(\{x\})$. Consider the mapping $\pi_1(M, x) \xrightarrow{\psi} \pi^{-1}(\{x\})$, constructed in Excercise 10.63, and $\psi(\gamma) = \Psi(\tilde{x}, \gamma)$. Show that it is a bijection.

**Exercise 10.65.** Show that $\pi_1(S^1) = \mathbb{Z}$.

**Exercise 10.66.** Show that $\pi_1((S^1)^n) = \mathbb{Z}^n$.

**Exercise 10.67 (*).** Show that for $(n > 1)$ one has $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$.

**Exercise 10.68.** Find the fundamental groups of all the letters of Greek alphabet, except $\Phi$ and $B$. (More precisely, graphs modelled by these letters.)

**Exercise 10.69 (*).** Given a finite connected graph with $n$ edges and $n$ vertices, consider its topological space $M$. Show that $\pi_1(M) = \mathbb{Z}$.

# GEOMETRY 11: Galois coverings

The subject of Galois covering that is covered in this exercise sheet is very similar to the Galois theory of field extensions. This is not a coincidence. In algebraic geometry methods from topology and differential geometry are applied to objects of algebro-geometric and number-theoretic nature. The version of Galois theory that is presented in ALGEBRA-11 goes back to A. Grothendieck. Grothendieck has given a definition of a fundamental group in such a way that Galois group and fundamental group of a topological space turned out to be particular cases of a more general construction. If one studies coverings and field extensions, it is very useful to keep in mind that these two things are similar.

All topological spaces in this exercise sheet are supposed to be Hausdorff.

**Exercise 11.1.** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and let $M_1$ be a connected component of $\tilde{M}$. Prove that $\pi^{-1}(M_1)$ is a connected component of $\tilde{M}$.

**Exercise 11.2 (!).** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and let $\tilde{M}$ and $M$ be connected and non-empty, and $\pi$ injective. Prove that $\pi$ is a homeomorphism.

**Definition 11.1.** Let $\tilde{M} \xrightarrow{\pi} M$, $\tilde{M}' \xrightarrow{\pi'} M$ be coverings. A **morphism of coverings** is a continuous map $\varphi: \tilde{M} \longrightarrow \tilde{M}'$, that respects the projection to $M$, in other words, such that $\varphi \circ \pi' = \pi$. The set of all morphisms between coverings is denoted by $\operatorname{Mor}(\tilde{M}, \tilde{M}')$. An isomorphism of covering is a morphism that is invertible, and moreover $\varphi^{-1} \circ \varphi = \mathsf{Id}$, $\varphi \circ \varphi^{-1} = \mathsf{Id}$.

**Exercise 11.3 (!).** Let $\varphi: \tilde{M} \longrightarrow \tilde{M}'$ be a morphism of coverings. Prove that $\varphi: \tilde{M} \longrightarrow \tilde{M}'$ is a covering.

**Exercise 11.4.** Let $M$ be connected and $\tilde{M} \xrightarrow{\pi} M$ be a covering. Prove that $\tilde{M}$ is locally connected.

**Exercise 11.5.** Let $M_1 \longrightarrow M_2$ and $M_2 \longrightarrow M_3$ be coverings.

** Is it true that the composition $M_1 \longrightarrow M_3$ is also a covering?

! Assume every point of $_3$ has a simply connected neighbourhood. Prove that $M_1 \longrightarrow M_3$ is a covering.

**Exercise 11.6.** Let $\tilde{M} \xrightarrow{\pi} M$, $\tilde{M}' \xrightarrow{\pi'} M$ be coverings and $\tilde{M}' \coprod \tilde{M}$ be their disjoint sum. Prove it is also a covering of $M$.

**Exercise 11.7.** Let $M$ be connected and $\tilde{M} \xrightarrow{\pi} M$ be a covering. Prove that $\tilde{M} \cong \coprod_{\alpha \in I} \tilde{M}_\alpha$ where $\{\tilde{M}_\alpha\}$ is the set of connected components of $\tilde{M}$ regarded as coverings of $M$.

**Definition 11.2.** A **splitting** of a covering $\tilde{M} \xrightarrow{\pi} M$ is an isomorphism between $\tilde{M}$ and a covering of the form $\tilde{M} \cong V \times M$ where $V$ is a set with discrete topologogy.

**Exercise 11.8.** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering of a connected space $M$. Prove that $\pi$ splits if and only if all connected components $\tilde{M}$ are isomorphic to $M$.

## Galois coverings

**Exercise 11.9 (!).** Let $M_1 \xrightarrow{\pi_1} M$, $M_2 \xrightarrow{\pi_2} M$ be coverings. Consider the following subset in $M_1 \times M_2$

$$M_1 \times_M M_2 := \{(m_1, m_2) \in M_1 \times M_2 \ \mid \ \pi_1(m_1) = \pi_2(m_2)\}$$

We consider $M_1 \times_M M_2$ as a topological space (with the topology induced from $M_1 \times M_2$). Prove that the natural map $M_1 \times_M M_2 \longrightarrow M$ is a covering.

**Definition 11.3.** The space $M_1 \times_M M_2$ together with the natural map to $M$ is called the **product of coverings $M_1$, $M_2$**. The product of arbitrary number of coverings is defined similarly.

**Remark.** If one uses the analogy between field extensions and coverings then disjoint unions of coverings correspond to a direct sums of semisimple Artinian rings, and products of coverings correspond to tensor products.

**Exercise 11.10.** Let $M_1$, $M_2$, $M_3$ be coverings of $M$. Proe that morphisms from $M_3$ to $M_1 \times M_2$ are in bijective correspondence with pairs of morphisms $\varphi_1 : \ M_3 \longrightarrow M_1$, $\varphi_2 : \ M_2 \longrightarrow M_1$.

**Exercise 11.11.** Consider $\mathbb{R}$ as a covering of $S^1$. How many connected components does $\mathbb{R} \times_{S^1} \mathbb{R}$ have?

**Definition 11.4.** Let $M_1 \xrightarrow{\varphi} M_2$ be a morphism between two coverings of . Define the **graph of** $\varphi$ as a subset in $M_1 \times_M M_2$ that consists of pairs of the form $(m, \varphi(m))$ for all $m \in M_1$.

**Exercise 11.12 (!).** Let $M_1 \xrightarrow{\varphi} M_2$ be a morphism between two coverings and let $\Gamma_\varphi$ be its graph. Prove that $\Gamma_\varphi$ is both open and closed in $M_1 \xrightarrow{\varphi} M_2$.

**Exercise 11.13.** Let $[\tilde{M} : M]$ be a covering, and moreover let $M$ and $\tilde{M}$ be connected (such a covering is called **connected**). Let $X \subset \tilde{M} \times_M \tilde{M}$ be a connected component. Prove that $X$ is the graph of an automorphism $\nu : \ \tilde{M} \longrightarrow \tilde{M}$ if and only if the projection on the first components is an isomorphism $X \cong \tilde{M}$.

**Exercise 11.14 (!).** Let $[\tilde{M} : M]$ be a connected covering. Consider the projection on the first argument $\tilde{M} \times_M \tilde{M} \longrightarrow \tilde{M}$ as a covering of $\tilde{M}$. Construct a bijective correspondence between $\operatorname{Mor}_{\tilde{M}}(\tilde{M}, \tilde{M} \times_M)$ and the set of automorphisms of $\tilde{M}$ over $M$.

**Hint.** Use the previous problem.

**Definition 11.5.** Let $[\tilde{M} : M]$ be a covering and assume $M$ and $\tilde{M}$ are connected. Then $[\tilde{M} : M]$ is called a **Galois covering**, if the covering $\tilde{M} \times_M \tilde{M} \longrightarrow \tilde{M}$ is split. In this situation the automorphism group of $\tilde{M}$ over $M$ is called the **Galois group of the covering** $[\tilde{M} : M]$ (denoeted $\operatorname{Gal}([\tilde{M} : M])$). Sometimes the Galois group is called **monodromy group**, or **deck transform group**.

**Exercise 11.15 (!).** Let $M$ be connected and let $[\tilde{M} : M]$ be such a Galois covering that every point of $M$ has exactly $n$ preimages (such a covering is called $n$-sheet covering). Prove that the Galois group $[\tilde{M} : M]$ has exactly $n$ elements.

**Hint.** Prove thta $[\tilde{M} \times_M \tilde{M} : \tilde{M}]$ is $n$-sheet covering too, and use the previous problem.

**Definition 11.6.** Let a group $G$ act on a set $S$. The action is called **free** if $s \neq gs$ for any $g \in G$, $s \in S$, if $g \neq 1$. The action is called **transitive** if for any two points $s_1$, $s_2 \in S$ there exists $g \in G$ such that $g(s_1) = s_2$.

**Exercise 11.16.** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and $G = \operatorname{Aut}_M(\tilde{M})$ be its automorphism group. Assume that $M$ is connected. Prove that for any $x \in M$ the group $G$ acts freely on $\pi^{-1}(x)$.

**Exercise 11.17 (!).** Let $\tilde{M} \xrightarrow{\pi} M$ be a Galois covering and let $x \in M$ be any point. Prove that $\operatorname{Gal}([\tilde{M} : M])$ acts on $\pi^{-1}(x)$ freely and transitively.

**Hint.** Find a bijective correspondence between $\pi^{-1}(X)$ and the set of connected componentns $\tilde{M} \times_M \tilde{M}$, and apply Exercise 11.14.

**Exercise 11.18 (!).** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and let $x \in M$ be any point. Prove that $\operatorname{Aut}_M(\tilde{M})$ acts transitively on $\pi^{-1}(X)$ if and only if $[\tilde{M} : M]$ is a Galois covering.

**Exercise 11.19.** Consider the covering $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$. Prove that it is a Galois covering.

**Exercise 11.20.** Take $n \in \mathbb{Z}$ and consider an $n$-sheeted covering $S^1 \longrightarrow S^1$, $t \mapsto nt$. Prove that this is a Galois covering.

**Definition 11.7.** Let $M$ be a topological space and let $G$ be a group that acts on $M$ by continuous transformations. Consider the space of $G$-orbits $M/G$. Recall (GEOMETRY-10) that the topology on $M/G$ is introduced a follows: a subset of $M/G$ is open if and only if its preimage in $M$ is open. The set $M/G$ with this topology is called a **quotient of** $M$ by the action of $G$.

**Exercise 11.21 (!).** Let $[\tilde{M} : M]$ be a covering and assume $G \subset \operatorname{Aut}_M(\tilde{M})$ acts on $[\tilde{M} : M]$ by automorphisms. Prove that this action is free and that the quotient $\tilde{M}/G$ is Hausdorff and is a covering of $M$.

**Remark.** Taking a quotient by the action of $G$ plays the same role in the Galois coverings theory as taking $G$-invariant in the Galois theory of field extensions.

**Exercise 11.22 (!).** Let $[\tilde{M} : M]$ be a covering and let $G$ be its automorphism group. Prove that $\tilde{M}/G$ is isomorphic to $M$ if and only if $[\tilde{M} : M]$ is a Galois covering.

**Hint.** Use Exercise 11.18.

**Remark.** In the several exercises that follow the statement and the proof mimic almost verbatim the corresponding exercises about Galois field extensions.

**Exercise 11.23.** Let $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3$ be a sequence of coverings, and moreover $\varphi_i$ are surjective and their composition is split. Prove that $\varphi_i$'s split.

By analogy with Galois theory of field extension the coverings of the form $\tilde{M} \xrightarrow{\pi} M$ will further be denoted $[\tilde{M} : M]$.

**Exercise 11.24 (!).** Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings, and assume all $M_i$ are connected and $[M_1 : M_3]$ is a Galos covering. Prove that $M_1 \times_{M_3} M_2$ splits as a covering of $M_1$.

**Hint.** Use Exercise 11.23, apply it to the sequence of coverings

$$M_1 \times_{M_3} M_1 \longrightarrow M_1 \times_{M_3} M_2 \longrightarrow M_1 \times_{M_3} M_3.$$

**Exercise 11.25 (!).** Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings and assume that $[M_1 : M_3]$ is a Galois covering. Prove that $[M_1 : M_2]$ is a Galois covering.

**Hint.** Use Exercise 11.23.

**Exercise 11.26.** Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings. Prove that

$$M_1 \times_{M_3} M_1 \cong M_1 \times_{M_2} (M_2 \times_{M_3} M_2) \times_{M_2} M_1.$$

**Exercise 11.27.** Deduce the following statement from the previous exercise: if $M_1 \longrightarrow M_2 \longrightarrow M_3$ is a sequence of coverings and if $[M_1 : M_2]$ and $[M_2 : M_3]$ is a Galois covering then $[M_1 : M_3]$ is also a Galois covering.

**Exercise 11.28.** Let $[\tilde{M} : M]$ be a covering and let $G$ be its Galois gorup and $G' \subset G$ be its subgroup. Consider the quotient $\tilde{M}/G'$. Prove that $[\tilde{M} : \tilde{M}/G']$ is a Galois covering with Galois group $G'$.

**Definition 11.8.** Let $\tilde{M} \longrightarrow M$ be a covering. A **quotient covering** $[\tilde{M} : M]$ is a covering $\tilde{M}' \longrightarrow M$ together with a sequence of coverings $\tilde{M} \longrightarrow \tilde{M}' \longrightarrow M$.

**Exercise 11.29 (!).** (fundamental theorem of Galois theory) Let $[\tilde{M} : M]$ be a Galois covering with Galois group $G$. Consider the correspondence that to a subgroup $G' \subset G$ associates a quotient covering $[\tilde{M}/G' : M]$. Prove that this correspondence defines a bijection between the set of subgroups of $G$ and the set of isomorphism classed of quotient coverings.

**Exercise 11.30.** Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings and assume that $[M_1 : M_3]$ is a Galois covering. Consider the natural projection

$$M_1 \times_{M_3} M_1 \xrightarrow{\Psi} M_2 \times_{M_3} M_2.$$

Let $g \in \operatorname{Gal}([M_1 : M_3])$ and $e_g \subset M_1 \times_{M_3} M_1$ be the graph $\{(m, g(m))\}$ of the action of $g$ in $M_1 \times_{M_3} M_1$. Prove that $g \in \operatorname{Gal}([M_1 : M_2]) \subset \operatorname{Gal}([M_1 : M_3])$ if and only if $e_g$ projects to the diagonal componont in $M_2 \times_{M_3} M_2$.

**Exercise 11.31.** Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of Galois coverings. Prove that the natural projection

$$M_1 \times_{M_3} M_1 \xrightarrow{\Psi} M_2 \times_{M_3} M_2.$$

defines a surjective homomorphism $\operatorname{Gal}([M_1 : M_3]) \xrightarrow{\psi} \operatorname{Gal}([M_2 : M_3])$. Prove that $\ker \psi = \operatorname{Gal}([M_1 : M_2])$.

**Hint.** Use the fact the Galois $\operatorname{Gal}([M_i : M_3])$ is identified with the set of connected components of $M_i \times_{M_3} M_i$ and use the previous problem.

**Exercise 11.32 (!).** Let $\tilde{M} \longrightarrow M$ be a Galois covering and $G' \longrightarrow \tilde{M}/G'$ be the bijective correspondence between quotient coverings and subgroups of the Galois gorup defined above. Prove that $G'$ is a normal subgroup if and only if $[\tilde{M}/G' : M]$ is a Galois covering.

## Coverings of linearly connected spaces

**Definition 11.9.** Let $M$ be a metric space. Recall that a **geodesic** in $M$ is a path $[a, b] \xrightarrow{\gamma} M$ such that $d(\gamma(x), \gamma(y)) = |x - y|$. The **length** of the geodesic is the distance between its ends. A path is called **piece-wise geodesic** if it can be decomposed into a union of a finite number of geodesic segments. The **length** of a piece-wise geodesic path is defined to be the sum of lengths of its geodesic pieces. We denote the length of a path $\gamma$ by $|\gamma|$.

**Definition 11.10.** Let $\Gamma$ be a graph and $M_\Gamma$ be its topological space. We say that $\Gamma$ is **connected** if its topological space is connected.

**Exercise 11.33 (!).** Prove that a graph is connected if and only if any two vertices are connected by a finite sequence of edges. Prove that a connected graph is linearly connected.

**Exercise 11.34 (!).** Let $\Gamma$ be a connected graph. By construction, on each edge $r_\alpha \subset M_\gamma$ of the graph are defined the coordinates that identify the edge with $[0,1]$. Let $\gamma$ be a piece-wise linear path in $\Gamma_M$, that is, a path that consists of a finite number of intervals of the form $[a_i, b_i] \xrightarrow{\varphi_i} [\lambda_i, \mu_i] \subset r_\alpha$, where $\varphi_i$ is linear. Define $|\gamma| := \sum |\lambda_i, \mu_i|$ as the sum of lengths of all intervals that contain this path. Define $d(x,y) := \inf |\gamma|$ where $\gamma$ runs throuh all piece-wise linear paths from $x$ to $y$. Prove that $d(x,y)$ defines a metric and $M_\Gamma$ is geodesically connected.

**Definition 11.11.** This metric is called the **standard metric on the topological space of a graph**.

**Definition 11.12.** Geodesically connected manifold $M$ is called **star-shaped** if any two points of $M$ are connnected by a unique geodesic.

**Exercise 11.35.** Prove that any convex subset in $\mathbb{R}^n$ (with the standard metric) is star-shaped.

**Exercise 11.36 (*).** Find a metric on $M = \mathbb{R}^2$ such that  is geodesically connected and there are infinitely many geodesics connecting arbitrary two fixed points.

**Exercise 11.37 (*).** Let $\Gamma$ be a tree, that is, a connected finite graph that has $n$ vertices and $n-1$ edges. Prove that $M_\Gamma$ with the standard metric is start-shaped.

**Exercise 11.38 (*).** Let $\Gamma$ be a finite graph such that $\Gamma_M$ is star-shaped. Prove that $\Gamma$ is a tree.

**Exercise 11.39 (!).** Let $M$ be a geodesically connected manifold, $\tilde{M} \xrightarrow{\pi} M$ be a covering, and $x$ and $y$ be two points in $\tilde{M}$. Consider the set $S_{x,y}$ of all paths on $\tilde{M}$ that connect $x$ and $y$, such that their projection to $M$ is piece-wise geodesic. Consider the following function on $\tilde{M} \times \tilde{M}$

$$\tilde{d}(x,y) = \inf_{\gamma \in S_{x,y}} |\pi(\gamma)|$$

Prove that it is a metric. Prove that $\tilde{d}(x,y) \geqslant d(\pi(x), \pi(y))$.

**Exercise 11.40 (*).** In the previous problem setting prove that $\tilde{M}$ is geodesically connected.

**Exercise 11.41.** Let $M$ be a geodesicall connected metric space and $\tilde{M} \longrightarrow M$ be its covering. Prove that the connected component of the preimage of a geodesic is a geodesic in $(\tilde{M}, \tilde{d})$.

**Hint.** Prove that the preimage of a geodesic is a geodesic in a neighbourhood of every point. Then use the inequality $\tilde{d}(x,y) \geqslant d(\pi(x), \pi(y))$.

**Exercise 11.42 (!).** Let $(M,d)$ be a star-shaped metric space and $\tilde{M} \xrightarrow{\pi} M$ be its connected covering. Let moreover $x \in \tilde{M}$ be any point and $U_x$ be the set of points $y \in M$ that can be connected with $x$ by a geodesic. Prove that $U_x$ is open and closed in $\tilde{M}$ and that $(U_x, \tilde{d})$ is star-shaped. Deduce that the natural projection $\tilde{M} \xrightarrow{\pi} M$ is an isometry and a homeomorphism.

**Hint.** Use the previous exercise.

**Exercise 11.43.** Let $M = [0,1]\times[0,1]$ be a square and $\tilde{M} \longrightarrow M$ be its connected covering. Prove that it is a homeomorphism.

**Exercise 11.44.** Let $M$ be a linearly connected and simply connected space, and $\tilde{M} \xrightarrow{\pi} M$ be a connected covering. Prove that it is a homeomorphism

**Hint.** Prove that $\tilde{M}$ is linearly connected. Let $x, y \in \pi^{-1}(x_0)$ be two points and $\tilde{\gamma}$ be a path that connects them. Then $\gamma := \pi(\tilde{\gamma})$ is a loop. Since $M$ is simply connected, $\gamma$ can be extended to a map from the square to $X \subset M$ (prove it). Consider the preimage of this square in $\tilde{M}$ and let $\tilde{X}$ be the component of the preimage that contains $\tilde{\gamma}$. Use the previous exercise to prove that $\tilde{X} \xrightarrow{\pi} X$ is a homeomorphism and deduces that $x = y$.

**Exercise 11.45.** In the previous problem setting prove that any covering $M$ splits.

**Definition 11.13.** Let $M$ be a any (not necessarily linearly connected) connected topological space. The space $M$ is called **simply connected** if any covering of $M$ is split.

**Remark.** Thanks to the previous exercise this definition is consistent with the definition of a simply connected linearly connected topological spaces given in GEOMETRY 10.

**Definition 11.14.** Let $M$ be connected. A covering $\tilde{M} \longrightarrow M$ is called **universal** if it is simply connected.

**Exercise 11.46 (!).** Prove a universal covering is a Galois covering.

**Exercise 11.47 (!).** Prove that universal covering is unique up to isomorphism.

**Hint.** Let $\tilde{M}$, $\tilde{M}'$ be two universally coverings of $M$. Since $\tilde{M} \times_M \tilde{M}'$ is a covering of $\tilde{M}$, $\tilde{M}'$, it splits over $\tilde{M}$, $\tilde{M}'$. This means that any connected component $\tilde{M} \times_M \tilde{M}'$ projects isomorphically to $\tilde{M}$, $\tilde{M}'$.

## Existence of the universal covering

**Exercise 11.48.** Let $M$ be linearly connected, $\tilde{M} \xrightarrow{\pi} M$ be a connected covering and $x \in M$ be any point. Prove that the cardinality of the set $\pi^{-1}(x)$ is not greater than the cardinality of $\pi_1(M)$.

**Exercise 11.49.** Prove that the cardinality $\pi^{-1}(x)$ is not greater than the cardinality of the set $M^{[0,1]}$ of maps from $[0,1]$ to $M$.

**Exercise 11.50 (*).** Let $\tilde{M} \xrightarrow{\pi} M$ be a connected covering of a connected $M$ and $x \in M$ be any point. Prove that the cardinality of $\pi^{-1}(x)$ is not greater than $|2^{2S}|$, where $|2^{2S}|$ is the cardinality of the set of subsets of $S \times S$.

**Hint.** Choose $x_1, x_2 \in \pi^{-1}(x)$. Prove that there exists a collection of such connected open subsets $\{\tilde{U}_\alpha\} \in \pi^{-1}(S)$ that $\tilde{U}_{\alpha_0}$ has non-empty intersection with the union of all $\tilde{U}_\alpha$ that are not equal to $U_{\alpha_0}$, and moreover

$$\{x_1, x_2\} = \pi^{-1}(x) \cap (\bigcup \tilde{U}_\alpha)$$

Decreasing the base $S$ if necessary one can assume that $\pi$ splits over $\pi(U_\alpha)$ for all $\alpha$. Prove that $x_2$ is determined uniquely if $x_1$, $\{\pi(U_\alpha)\}$ is given, and if it is known which $U_\alpha$ have non-empty intersection.

**Exercise 11.51.** Let $M$ be connected and let $V$ be a set of a cardinality defined below. Denote by $\mathcal{R}$ the set of all topologies defined on some subset $X \subset M \times V$ in such a way that the only natural projection $X \longrightarrow M$ is a covering. Prove that any connected covering $M$ is isomorphic to some element of $\mathcal{R}$ if

a. $M$ is linearly connected and the cardinality of $V$ is $|M^{[0,1]}|$

b. **(*)** The cardinality of $V$ is $|2^{2^S}|$ where $S$ is the base of topology on $M$.

**Remark.** This exercise allows one to speak of "the set of isomorphism classes of coverings". Recall that not all mathematical objects are sets; thus, the class of all sets is not a set. In order to prove that a class is a set, one has to restrict its cardinality.

**Definition 11.15.** Let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be a collection of maps onto $M$ indexed b $I$ (possibly infinite, and even uncountable). Consider the set of all $(m_{\alpha_1}, m_{\alpha_1}, \dots) \in \prod M_\alpha$ such that $\pi_\alpha(m_\alpha) = m$ for some $m \in M$. This set is called the **fibre product of** $\{M_\alpha\}$ and is denoted by $\prod_M M_\alpha$.

**Exercise 11.52.** Let $M$ be a topological space and $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be a collection of its coverings. Introduce on $\prod_M M_\alpha$ a topology in the following way. Let $U \subset M$ be open and let $\{U_\alpha \subset M_\alpha\}$ be the collection of open sets that cover $U$. Prove that the sets of the form $\prod_U U_\alpha \subset \prod_M M_\alpha$ define the base of topology on $\prod_M M_\alpha$. Prove thta $\prod_M M_\alpha$ is Hausdorff.

* Is it true that the natural projection $\prod_M M_\alpha \longrightarrow M$ is a covering?

! Suppose that every point of  has a simply connected neighbourhood. Prove that the natural projection $\prod_M M_\alpha \longrightarrow M$ is a covering.

**Definition 11.16.** In this situation $\prod_M M_\alpha$ is called the **fibred product of** $M_\alpha$ **over** $M$ or just a product of coverings $M_\alpha \xrightarrow{\pi_\alpha} M$.

**Exercise 11.53.** Assume all coverings $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ are split. Prove that $\prod_M M_\alpha$ split too.

**Exercise 11.54 (!).** Let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be a Galois covering. Prove thta any connected component of their product over $M$ is a Galois covering too.

**Hint.** Use the Exercise 11.53.

**Exercise 11.55.** Let $\tilde{M}$ be a covering of $M$. Constructi the natural bijection between $\operatorname{Mor}(\prod_M M_\alpha, \tilde{M})$ and $\prod \operatorname{Mor}(M_\alpha, \tilde{M})$

**Exercise 11.56 (*).** Let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be the set of all coverings $S^1 \longrightarrow S^1$, $t \longrightarrow nt$, indexed by $n \in \mathbb{Z}$. Prove that any connected component of $\prod_M M_\alpha$ is isomorphic to $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1$.

**Exercise 11.57.** Let $\tilde{M} \xrightarrow{\pi} M$ be a covering, and assume that $\tilde{M}$ and $M$ are connected, $x \in M$, $x_1, x_2 \in \pi^{-1}(x)$, $W$ is the connected component of $\tilde{M} \times_M \tilde{M}$ that contains $x_1 \times x_2$, and $W_1$ is the connected component of $\tilde{M} \times_M \tilde{M} \times_M \tilde{M}$, that contains $x_1 \times x_2 \times x_2$. Prove that the natural projection $W_1 \longrightarrow W$ (forgetting the third argument) is an isomorphism.

**Exercise 11.58.** In the same situation, let $\{x_\alpha\}$ be a set of points in $\pi^{-1}(x)$, indexed by $\alpha \in I$, and let $W$ be the corresponding component in the fibre product $\prod_{M,I} \tilde{M}$ of $I$ copies of $\tilde{M}$, and $W_1$ be a component in $\left(\prod_{M,I} \tilde{M}\right) \times_M \tilde{M}$, that contains $\{x_\alpha\}$ and $x_0$, and moreover $x_0 \in \{x_\alpha\}$. Prove that the natural projection $W_1 \longrightarrow W$ is an isomorphism.

**Exercise 11.59 (!).** Let $\tilde{M} \xrightarrow{\pi} M$ be a connected covering and $x \in M$. Consider the product $\prod_{M,\{\pi^{-1}(x)\}} \tilde{M}$ of $\tilde{M}$ with itself indexed by the set $\pi^{-1}(x)$, and let $\tilde{M}_G$ be the connected component in $\prod_{M,\{\pi^{-1}(x)\}} \tilde{M}$ containing the product of all $x_\alpha \in \{\pi^{-1}(x)\}$. Prove that $\tilde{M}_G \times_M \tilde{M}$ splits over $\tilde{M}_G$. Prove that $\tilde{M}_G \longrightarrow M$ is a Galois covering.

**Remark.** We have proved that any covering is a quotient covering of a Galois covering.

**Exercise 11.60.** Let $M$ be a connected topological space, $\mathcal{R}$ be the set of all isomorphism classes of connected coverings of $M$, and let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be the corresponding set of coverings, and $\tilde{M} \subset \prod_M M_\alpha$ be the connected component of their product. Prove that for any connected covering $\tilde{M}' \longrightarrow M$ there exists a surjective morphism of coverings $\tilde{M} \longrightarrow \tilde{M}'$.

**Hint.** Use the previous exercise.

**Exercise 11.61.** In the previous problem setting prove that $\tilde{M}$ is a Galois covering.

**Exercise 11.62 (!).** Deduce tht for any $\tilde{M} \longrightarrow M$ the covering $\tilde{M} \times_M \tilde{M}' \longrightarrow \tilde{M}$ splits.

**Hint.** Use the Exercise 11.24.

**Exercise 11.63 (!).** Let $M$ any connected topological space, $\tilde{M} \longrightarrow M$ be a Galois covering constructed above. Prove that $\tilde{M}$ is simply connected.

**Remark.** We have obtained that any connected topological space has a universal covering. As was shown above, the universal cover is unique.

**Exercise 11.64 (!).** Let $M$ be linearly connected, and $\tilde{M}$ be its universal covering, and $\operatorname{Gal}([\tilde{M} : M])$ be the corresponding Galois group. Prove that $\operatorname{Gal}([\tilde{M} : M])$ is not isomorphic to the Galoi group of $M$.

**Definition 11.17.** The **fundamental group** of a topological space  is the group $\pi_1(M) := \operatorname{Gal}([\tilde{M} : M])$, where $\tilde{M}$ is the universal covering.

**Definition 11.18.** Subgroups $G_1, G_2 \subset G$ are called **conjugated** if there exists $g \in G$ such that $G_1$ is mapped to $G_2$ by the automorphism $x \longrightarrow x^g$.

**Exercise 11.65 (*).** Let $M_1 \longrightarrow M$ be a covering, and let $\tilde{M} \longrightarrow M_1 \longrightarrow M$ be the universal covering. Consider the subgroup $G_1 \subset \operatorname{Gal}([\tilde{M} : M]) = \pi_1(M)$, obtained as a result of the fundamental theory of the Galois theory. Prove that this correspondence defines a bijection between isomorphism classes of coverings of $M$ and conjugacy classes of subgroups of $\pi_1(M)$.

**Exercise 11.66 (!).** Find all coverings of a circle up to isomorphism. Construct them explicitly.

**Exercise 11.67 (*).** Let $M$ be a connected topological space such that all linear connected components of it are simply connected. Can it have a non-trivial fundamental group?

**Exercise 11.68 (*).** Let $B$ be the set of polynomials $P(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_0$ over $\mathbb{C}$ that have distinct roots and let $B_1$ be the set of all tuplese $(x_1, \ldots, x_n) \in \mathbb{C}^n$ of pairwise distinct numbers $x_i \in \mathbb{C}$. Introduce on $B$ and $B_1$ the natural topology of a subset of $\mathbb{C}^n$. Consider the map $B_1 \xrightarrow{\pi} B$, $(x_1, \ldots, x_n) \longrightarrow \prod(t - x_i)$. Prove that $\pi$ is a Galois covering. Find its Galois group.

**Exercise 11.69 (*).** Construct a connected covering that is not a Galois cover.

# GEOMETRY 12: fundamental group and homotopies

## Homotopies

All topological spaces in this exercise sheet are assumed locally arcwise connected and Hausdorff, unless the contrary is stated.

**Definition 12.1.** Let $f_1, f_2 : X \longrightarrow Y$ be a continuous map of topological spaces. Recall that a **homotopy** between $f_1$ and $f_2$ is a continuous map $F : [0,1] \times X \longrightarrow Y$ such that $F\Big|_{\{0\}\times X}$ equals $f_1$, and $F\Big|_{\{1\}\times X}$ equals $f_2$.

**Exercise 12.1.** Prove that maps that are homotopic induce the same morphism $\pi_1(X) \longrightarrow \pi_1(Y)$.

**Definition 12.2.** Let $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ be continuous maps of topological spaces, moreover, $f \circ g$ $g \circ f$ are homotopic to identity maps from $X$ to $X$ and from $Y$ to $Y$. Such maps are called **homotopy equivalences** and $X$ and $Y$ are then called **homotopy equivalent**.

**Exercise 12.2.** Prove that a composition of homotopy equivalence between maps is a homotopy equivalence. Prove that a homotopy equivalence of spaces is an equivalence relation.

**Exercise 12.3 (!).** Let $f : X \longrightarrow Y$ be a homotopy equivalence. Prove that $f$ induces an isomorphism of fundamental groups.

**Exercise 12.4.** Let $X \subset Y$ be a retraction. Prove that $X$ are $Y$ homotopy equivalent.

**Exercise 12.5 (!).** Let $X$ be a topological spaces. Prove that $X$ is contractible if and only if it is homotopy equivalent to a point.

**Exercise 12.6 (!).** Consider the connected graph $\Gamma$ which has $n$ edges and $n$ vertices. Prove that the associated topological space is homotopy equivalent to a circle.

**Exercise 12.7 (!).** Let $M$ be a connected topological space and let $x, x', y, y' \in M$ be any points. Prove that the spaces of paths $\Omega(M, x, x')$ and $\Omega(M, y, y')$ are homotopy equivalent.

**Hint.** Consider a path $\gamma_{xy}$ that connects $x$ and $y$ and let $\gamma_{x'y'}$ be the path that connects $x'$ and $y'$. Let $\gamma_{xy}^{-1}(t) = \gamma_{xy}(1-t)$ and $\gamma_{x'y'}^{-1}(t) = \gamma_{x'y'}(1-t)$. Consider the map $f : \Omega(M, x, x') \longrightarrow \Omega(M, y, y')$ that maps any path $\gamma \in \Omega(M, x, x')$ into the composition $\gamma_{xy}^{-1}\gamma\gamma_{x'y'}$, and the analogous map $g : \Omega(M, y, y') \longrightarrow \Omega(M, x, x')$ that maps $\gamma \in \Omega(M, y, y')$ to $\gamma_{xy}\gamma\gamma_{x'y'}^{-1}$. Prove that $fg$ is homotopic to the identity maps and that $gf$ is homotopic to the identity map.

## The space of paths on locally contractible spaces

**Definition 12.3.** Let $M$ be a topological space. The space $M$ is called **locally contractible** if every point has a contractible neighbourhood.

**Exercise 12.8.** Let $M$ be a locally contractible topological space. Prove that $M$ is locally arcwise connected.

**Exercise 12.9 (*).** Let $M$ be a geodesically connected metric space such that for some $\delta > 0$ any two points that are at a distance $< \delta$ one from another are connected by a unique geodesic. Prove that $M$ is locally contractible.

**Exercise 12.10.** Prove that any graph is locally contractible.

**Definition 12.4.** A topological space $M$ is called a **manifold of dimension** $n$ if any point has a neighbourhood that is homeomorphic to an open ball in $\mathbb{R}^n$.

**Remark.** Manifolds are easily seen to be locally contractible.

**Exercise 12.11 (!).** Prove that a sphere $S^n$ is a manifold.

**Hint.** Use the stereographic projection.

**Exercise 12.12.** Let $M$ be contractible, $x, y \in M$. Prove that all paths $\gamma \in \Omega(M, x, y)$ are homotopic.

**Exercise 12.13 (!).** Let $\gamma \in \Omega(M, x, y)$ be a path in a locally contractible space $M$ and $\{U_\alpha\}$ be the set of contractible open sets in $M$. Choose a finite set in $\{U_\alpha\}$ such that it covers $\gamma$ (this can be done since $\gamma$ is compact). Let $V_1, \dots, V_n$ be the corresponding cover of $[0, 1]$ with connected intervals where every $V_i$ is a connected component of $\gamma^{-1}(U_i)$, and all $U_i$ are contractible. Order $V_i$ in such a way that $V_i$ and $V_{i+1}$ intersect at a point $t_i$, and let $a_i := \gamma(t_i)$. Prove that any path $\gamma' \in \Omega(M, x, y)$ such that $\gamma'(t_i) = a_i$, and $\gamma'([t_i, t_{i+1}]) \subset U_i$, is homotopic to $\gamma$.

**Hint.** Use the previous exercise.

**Exercise 12.14 (!).** Let $M$ be a locally contractible topological space, and $\gamma \in \Omega(M, x, y)$ is a path. Prove that $\gamma$ has a neighbourhood $\mathcal{U} \subset \Omega(M, x, y)$ such that all $\gamma' \in \mathcal{U}$ are homotopic.

**Hint.** Use the previous problem.

**Remark.** Notice that on all compact manifolds of dimension $> 1$ there are loops that are defined by a surjective map. Such loops can be constructed in the same way as the Peano curve.

**Exercise 12.15 (!).** Let $M$ be a manifold (for instance, a sphere) of dimension greater than 1, and $\gamma \in \Omega(M, x)$ be a loop. Prove that $\gamma$ is homotopic to a loop that is not surjective.

**Hint.** Use the previous exercise.

**Exercise 12.16 (!).** Let $n > 1$. Prove that $n$-dimensional sphere is simply connected.

**Hint.** Let $\gamma$ be a loop on a sphere. Use the previous exercise and find a homotopy from $\gamma$ to a loop that maps $[0, 1]$ to $S^n \backslash \{x\}$ where $x$ is some point. Prove that a sphere without a point is homeomorphic to $\mathbb{R}^n$, and in particular is contractible.

**Exercise 12.17 (*).** Let $M$ be contractible and let $F: M \times [0, 1] \longrightarrow M$ be a homotopy from the identity map to the constant map $M \to y \in M$. Consider the following map $M \longrightarrow \Omega(M, y, *)$, $t, m \longrightarrow F(m, t)$ ($t \in [0, 1]$, $m \in M$). Prove that it is continuous.

**Exercise 12.18.** Let $M$ be locally contractible, $x, y \in M$ be two points and $\gamma \in \Omega(M, x, y)$ be a path. Prove that $\gamma$ has a neighbourhood $\mathcal{U} \in \Omega(M, x, *)$, such that all paths $\gamma' \in \mathcal{U}$ that connect $x$ and $a$ are homotopic in $\Omega(M, x, a)$.

**Exercise 12.19 (*).** Let $M$ be a locally contractible topological space, and let $x \in M$ be a point, and $\Omega(M, x, *)$ be the set of all paths that start at the point $x$ endowed with the compact-open topology. Consider the equivalence relation on $\Omega(M, x, *)$: $\gamma \sim \gamma'$ if $\gamma$ and $\gamma'$ connect $x$ and $y$, and homotopic in $\Omega(M, x, y)$. Consider $\Omega(M, x, *)/\sim$ with the quotient topology. Consider a contractible neighbourhood $U_y \ni y$, and let $U_y \xrightarrow{F} \Omega(U_y, y, *)$ be a mapping that was constructed in the exercise 12.17. Let $\gamma \in \Omega(M, x, y)$ be a path and $U_y \xrightarrow{\Psi} \Omega(M, x, *)$ be a mapping that maps $a \in U_y$ to a path $\gamma F(a)$ (that is, to a path that is defined on $[0, 1/2]$ as $t \longrightarrow \gamma(2t)$, and on $[1/2, 1]$ as $F(a, 2t-1)$. Prove that (for sufficiently small$U_y$) $\Psi$ composed with $\Omega(M, x, *) \xrightarrow{\pi} \Omega(M, x, *)/\sim$ is a homeomorphism $U_y$ on some open subset in $\Omega(M, x, *)/\sim$.

**Hint.** Continuity of $\Psi \circ \pi$ is obvious by construction and injectivity follows from the previous exercise. In order to show that $\Psi \circ \pi$ defines a homeomorphism $U_y$ on $\Psi \circ \pi(U_y)$ we need to show that prove $\Psi \circ \pi$ maps open sets to open sets. This is clear from the fact that the natural map $\Omega(M, x, *)/\sim \longrightarrow M$, $\gamma' \longrightarrow \gamma'(1)$ is continuous and defines a homeomorphism $U_y$ on its image.

**Exercise 12.20 (*).** Consider the mapping $\Omega(M, x, *)/\sim \longrightarrow M$ that maps a path $\gamma \in \Omega(M, x, y)$ to the point $y = \gamma(1)$. Prove that this is a covering.

**Hint.** Use the previous exercise.

**Exercise 12.21 (!).** Prove that $\Omega(M, x, *)$ is contractible.

**Exercise 12.22 (*).** Prove that $\gamma$ is a path in $\Omega(M, x, *)/\sim$. Prove that $\gamma$ is homotopic to an image of some path from $\Omega(M, x, *)$.

**Hint.** Prove that $\gamma$ can be lifted to a path in $\Omega(M, x, *)$ locally and use the fact that for any point in $\Omega(M, x, *)/\sim$ its preimage in $\Omega(M, x, *)$ is connected.

**Exercise 12.23 (*).** Deduce that $\Omega(M, x, *)/\sim$ is simply connected.

**Remark.** Let $(M, x)$ be a locally connected topological space with a marked point. The universal covering of $M$ can be thus identified with the set of pairs ($y \in M$, homotopy class of a path $\gamma \in \Omega(M, x, y)$).

## Free group and wedge sum

**Definition 12.5.** Let $(M_1, x_1)$, $(M_2, x_2)$, $(M_3, x_3)$, ... be a collection (possibly infinite) of connected topological spaces with a marked point. Consider the quotient space of a disconnected sum of all $(M_\alpha, x_\alpha)$ by the equivalence relation $\{x_1\} \sim \{x_2\} \sim \{x_3\} \sim \ldots$ This quotient space is called a **wedge sum**, denoted by $\bigvee_\alpha (M_\alpha, x_\alpha)$. A wedge sum can also be denoted by $(M_1, x_1) \vee (M_2, x_2) \vee (M_3, x_3) \vee \ldots$

**Exercise 12.24.** Assume that all $M_\alpha$ are connected (arcwise connected, Hausdorff). Prove that the wedge sum is connected (arcwise connected, Hausdorff).

**Exercise 12.25 (!).** Assume that all $M_\alpha$ are connected and simply connected. Prove that their wedge sum is simply connected.

**Exercise 12.26 (!).** Let $\Gamma$ be a connect graph that has $n$ vertices and $n+k-1$ edges. Prove that its associated topological space $M_\Gamma$ is homotopy equivalent to a wedge sum of $k$ circles.

**Hint.** Assume $\Gamma$ has an edge $r$ that connects two distinct vertices $v_1, v_2$. Consdireth graph $\Gamma'$ with $n-1$ vertices and $n+k-2$ edges that is obtained from $\Gamma$ in the following way. Remove an edge $r$ from $\Gamma$ and glue vertices $v_1$ and $v_2$ together. Prove that $M_\Gamma$ and $M_{\Gamma'}$ are homotopy equivalent.

**Definition 12.6.** Consider a set $\{a_1, a_2, \ldots\}$ of cardinality $N$ ($N$ by either finite or infinite). An $N$**-ary tree** $D_N$ is an infinite graph that is defined in the following way. The verticesof $D_N$ are finite sequences of symbols $a_i$. The edges connect vertices that correspond to $A_1 A_2 \ldots A_k$ and $A_1 A_2 \ldots A_k A_{k+1}$ (all $A_i$ belong to $\{a_1, a_2, \ldots\}$).

**Exercise 12.27.** Prove that every vertex $D_N$ has $N+1$ incoming edges.

**Exercise 12.28 (!).** Let $M_N$ be a topological space of an $N$-ary tree, with the natural metric, defined in the beginning of this exercise sheet. Prove that $M_N$ is star-shaped (any two points can be connected by a unique geodesic). Prove that it is contractible.

**Exercise 12.29 (!).** Consider an $2N-1$-ary tree. Colour its edges in $N$ colours in such a way that any vertex has 2 incoming edges of each colour. Consider the wedge sum of $N$ circles and colour each of the circles in a diffferent colour. Consider the mapping from $M_{2N-1}$ to the wedge sum of $N$ circles that maps the vertices of the graph to the vertices of the wedge sum and an edge of colour $a_i$ to the circle of the same colour. Prove that this is a universal cover.

**Exercise 12.30.** Let $\{a_1, a_2, \ldots\}$ be a set of cardinality $N$, and let $\mathcal{W}$ be the set of finite sequences ("words") of symbols $a_i$, $a_j^{-1}$, such that subsequences of the form $a_i a_i^{-1}$ and $a_i^{-1} a_i$ never occur. A sequence of length 0 is denoted $e$. We multiply words by juxtaposing them and striking out all $a_i a_i^{-1}$, $a_i^{-1} a_i$ that might occur. Prove that $\mathcal{W}$ forms a group.

**Definition 12.7.** This group is called the **free group generated by** $\{a_1, a_2, \ldots\}$ and is denoted $F_N$.

**Exercise 12.31.** Prove that $F_1$ is isomorphic to $\mathbb{Z}$.

**Exercise 12.32 (!).** Let $G$ be a group and $\{g_1, g_2, \ldots\}$ be a collection of elements from $G$, labelled $\{a_1, a_2, \ldots\}$. Prove that there exists a unique homomorphism $F_N \longrightarrow G$ that maps $a_i$ to $g_i$.

**Exercise 12.33 (!).** Find a free action of $F_N$ on the topological space $M_{2N-1}$ of an $2N-1$-ary tree that is transitive on vertices.

**Exercise 12.34 (!).** Prove that $M_{2N-1}/F_N$ is a wedge sum of $N$ circles and that the fundamental group of the wedge sum is free.

**Exercise 12.35 (!).** Prove that any (possibly infinite) graph is homotopy equivalent to a wedge sum of circles.

**Exercise 12.36 (!).** Deduce that any subgroup of a free group is free.

**Hint.** Use the Galois theory of coverings.

**Exercise 12.37 (*).** Let $G_1, G_2, \dots$ be a set of groups. Consider the set $\mathcal{W}$ of finite sequences of non-identity elements from different $G_i$ such that elements of the same group never occur next to each other. Given any sequence $A$ of elements from $G_i$ one can obtain an element $\mathcal{W}$ the following way. If $A$ has to successive elements from $G_i$, we multiply them and replace these elements with their product. If $A$ an identity element of one of the groups we strike it out. Repeat this procedure as many times as needed in order to get an element from $\mathcal{W}$. The elements of $\mathcal{W}$ can be multiplied by juxtaposing words and applying the procedure above. Prove that this defines a group.

**Definition 12.8.** This group is called the **free product of groups** $G_1$**,** $G_2$**,** $\dots$.

**Exercise 12.38.** Prove that the free group on $N$ generators is the free product of $N$ copies of $\mathbb{Z}$.

**Exercise 12.39.** Prove that a free product of free groups is free.

**Exercise 12.40 (*).** Let $(M_1, x_1), (M_2, x_2), (M_3, x_3), \dots$ be a collection of connected topological spaces with a marked point. Prove that $\pi_1(\bigvee_\alpha (M_\alpha, x_\alpha))$ is isomorphic to a free product of groups $\pi_1(M_1, x_1), \pi_1(M_2, x_2), \pi_1(M_3, x_3), \dots$.